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De Sitter Quantum Mechanics and Inflationary Matrix Cosmology of Self-tuned Universe

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We observe that the large N world sheet RG in $c = 1$ matrix model, formulated in [1, 2], with N^2 quantum mechanical degrees of freedom at small compactification radius is capable of capturing dimensional mutation. This manifests in deforming the familiar AdS_2 quantum mechanics in the minisuperspace Wheeler-de Witt (WdW) cosmology of the $2D$ quantum gravity, obtained by the large N RG with N quantum mechanical degrees of freedom only, to a modified WdW cosmology describing tunneling to an inflationary de Sitter vacuum and its evolution. The scale fluctuation plays an important role in providing an ansatz for uniquely choosing the initial wave function. We observe that the nonperturbative effects due to the N^2 quantum mechanical degrees of freedom introduce explicit open string moduli dependence in the wave function via the Hubble scale, which determines the geometry of the true vacuum one tunnels to. The modified WdW equation also captures controlled formation of baby universes of vanishing size that self-tunes the de Sitter cosmological constant to be small positive.

Contents

1 Introduction

One of the central problems in quantum gravity in recent time is to find a vacuum wave function for a consistent quantum mechanical description of the early universe (with respect to the recent observational data) and to choose it uniquely, if there is a dynamical selection principle.

A simple quantum cosmological approach [3] (see [4] for a review) would be to arrive at a Schrödinger like (WdW) equation that would describe tunneling of the universe from *nothing* to an inflationary de Sitter vacuum [5], rather than to an Euclidean Hartle-Hawking vacuum [6]. Once the schrödinger equation is derived, one would also like to have its unique solution [7]. We refer to [8, 9, 10, 11, 12, 13, 14, 15] for a discussion on quantum creation of inflationary universe from nothing. One can see recent works, for example [16] and references therein, for a comparative discussion of the two types of wave functions in string theory context.

In string theory the universe can be chosen to be the holes in the world sheet, that can be filled with D-branes. The dimensionality of the universe and the issue of openness/closedness or the homogeneity/inhomogeneity depends on how we fill these holes. Even though these holes are world sheet quantities, they can have a good space-time description. The one point function of these objects describes the wave function ψ of the universe with a scale factor $l = \oint e^{\frac{1}{2}\gamma\varphi} d^2\sigma$ (φ is the Liouville field in a noncritical theory). In minisuperspace approximation, the WdW equation (a BRST constraint of the field theory)

$$\frac{1}{2}(L_0 + \bar{L}_0)|\psi\rangle = 0 \tag{1.1}$$

is solved for Euclidean Hartle-Hawking vacuum with homogenous FRW cosmology. Because of reaching a huge size in a small time interval, for most of the purposes the cosmological evolution of the inflationary universe is semiclassical and minisuperspace computations successfully capture a lot of features [19]. However, in an inflationary universe, the long wave fluctuations of the scalar field drives geometry at much larger scales (at $l \gtrsim H^{-1} \sim M_p^{-1}$) to be highly inhomogeneous, far away from that of a homogeneous Friedmann space. The quantum fluctuation of the scalar field may drive some parts of the universe to inflate eternally, whereas other parts pass the end of inflation and undergo different phases of evolution. Much of the important inflationary physics are actually lost in the realm of the minisuperspace approximation, that can not get rid of an initial presumption of global homogeneity of the universe. The initial wave function in early universe dealing a cosmological singularity with particle production on the horizon is essentially quantum mechanical and is worth seeking a proper stringy description (see [20] for a review on the subject; also see [21] for a recent study of decay of metastable vacua in $2D$ Liouville gravity, for issues on nucleation in $2D$ Liouville gravity).

The problem can be defined in the following way. The cosmological singularities often can be resolved by a phase of suitably chosen closed string tachyon condensate much like the exponential wall in a time-like Liouville theory (see [22, 23] and references therein). Generically the fields in such tachyon phase become heavy and tend to evacuate configurations of real or virtual particles or excitations that source any component of string field (see [24] and further references). More specifically the S-matrix of the system, that governs the perturbative $BRST$ consistency condition on the states in the tachyon condensate phase, does not have perturbative asymptotic particle poles. As a result string world sheet reaches spatial infinity in a finite world sheet time, giving rise to holes in the world sheet. Though in special cases D-brane boundary states make these holes consistent, generically they give rise to $BRST$ anomaly that can be resolved by considering correlations of these holes corresponding to their unitary mapping. These correlations can be nonlocal representing interaction between multiple world sheets [25, 26] and can be complicated enough to model the dynamics in the black hole interior (see examples discussed in [24]). Note that the above scenario of preserving unitarity is reminiscent of the black hole final state proposal for solving the information paradox [27]. However, imposing cancelation of $BRST$ anomaly alone does not specify a unique correlation and thus a unique final state, rather yields a restricted set of states in the tachyon condensate phase.

In the light of above discussion the vacuum wave function for the early universe can be described by a wave function for some kind of tachyon condensate $\langle T \rangle$ that replaces the initial cosmological singularity. A simplest undeformed condensate would correspond to the Euclidean Hartle-Hawking vacuum whose evolution is given by the usual WdW equation [1]. However, the stringy modification to the WdW equation can come from an anomaly in the BRST like constraint (1.1) that is resolved by a nontrivially deformed condensate $\langle T + \delta T \rangle$ (See [30, 31, 32, 33] for other relevant approaches/proposals of modifying the Hartle-Hawking vacua). The allowed set of states surviving in this nontrivial deformed tachyon condensate phase can arise from various 'beyond minisuperspace' contributions. As we discussed before, they can arise from nonlocal correlations of the holes via multi-trace [25, 26] or topology changing amplitudes that are essentially beyond minisuperspace [4]. These amplitudes describe emission or absorption of baby universes in the WdW cosmology [34, 35, 36, 37] that can self-tune the cosmological constant to appropriate value. However, in a full nonperturbative setup (like the one we are going to use in this paper) the deformed condensate can contain nonperturbative objects that can feel all the nonzero modes of dilaton. Note that, in the perturbative computation of Euclidean vacuum in [28], the tachyon condensation effectively masses up all the closed string modes and hence the fluctuations in dilaton too, freezing it to its bulk weakly coupled value. Thus the setup remains perturbative there. There are other similar evidences of perturbativity in the closed string tachyon condensation such as repulsion felt by D-brane probe from winding tachyon phase [38] (also see fate of twisted D-branes in orbifolds [39, 40]).

However, in this work we will argue that the presence of nonperturbative objects, that penetrate the tachyon wall down to strong coupling (also see the $2D$ example in [41]), make it possible to feel the dilaton fluctuations which play a vital role in uniquely determining the

[1] In time dependent background in perturbative string theory (in type IIB and as well as in heterotic strings) such a condensate is constructed from the correlators supported in the weakly coupled bulk [28] that seems to be a useful setup to study space-like big bang singularities in Milne space-time. For a nonperturbative study, see [29].

quantum state corresponding the nontrivially deformed tachyon condensate. We will also discuss how the ability to feel the dilaton fluctuations can have implications in resolving a cosmological singularity. In particular we note the possibility of using the properties of density perturbation spectra, such as its near flatness, to determine an ansatz for dynamical vacuum selection via the initial profile of the scale (dilaton) fluctuation.

In the language of WdW cosmology, computing the wave function for the appropriate deformed tachyon condensate replacing a cosmological singularity would require going beyond the minisuperspace by considering general matter dependent deformations and deformations due to higher order derivatives in scale. In other words one needs to access the full superspace by considering the arbitrary scale and matter fluctuations due to all the nonzero modes of dilaton and matter. The nonzero modes of matter can be excited by appropriately considering Neumann boundary condition on the world sheet holes. However, the role of arbitrary matter fluctuations in a nonperturbative tachyon condensate we are considering, is an interesting issue that we take up elsewhere [42]. Thus studying an initial wave function for an inflationary vacuum eventually would be tied to a time dependent nonperturbative setup capturing dynamical fluctuations and nonlocal processes. Even though we start with a (sub)critical background, we believe the phenomena we are observing is generic due to dimensional mutation brought by these nonperturbative objects, that we explain below.

So far we focussed on the question of capturing the inflationary nature of the initial wave function and its unique solution. Now the issue of getting a (Lorentzian) de Sitter vacuum as opposed to an (Euclidean) AdS is tied to the question of accessing the supercritical regime of the theory instead of a (sub)critical one [35, 36, 37]. Let us take up the example of bosonic string theory. In $2D$ noncritical bosonic string theory (Liouville coupled with $c = 1$ matter) the minisuperspace WdW equation for $\langle T \rangle$, both in the continuum and discrete version (free fermionic picture of matrix quantum mechanics) is a Bessel equation solving for an Euclidean AdS_2 vacuum ($\psi \sim K_0(\sqrt{\mu} l)$, $\mu > 0$) (see the reviews [43, 44, 45] and [46]). The minisuperspace equation in a supercritical setup (Liouville coupled to $c > 25$) with a tachyon condensate in one of the matter directions solves for global de Sitter in the far future ($H_0(\sqrt{-\mu} l)$) [46]. However, due to the simplicity of the condensate, the vacuum wave function does not capture any kind of inflationary scenario or proliferation of the de Sitter. The striking feature is that, the minisuperspace WdW equation in this case is apparently a de Sitter quantum mechanics of single matter but with other $c = 25$ matter directions fixing the Weyl anomaly. Thus their only role is in fixing the de Sitter sign of the cosmological constant, the Lorentzian signature of the world sheet metric and a sensible interdependence of the scale and the string coupling for a viable cosmological scenario. On the other hand, remnants of dS_2 quantum mechanics is already there even in minisuperspace $2D$ noncritical theory. This is evident from the free fermion quantum mechanics, where a special class of time (or $c = 1$ matter) dependent w_∞ deformations of the

Fermi sea gives rise to past and future space-like boundaries like a space-time cosmological event horizon [47] [2]. Let us here remind ourselves that the semiclassical condensate $\langle T \rangle$ giving rise to Euclidean Hartle Hawking vacuum (AdS_2) of the $c = 1$ theory, can be thought of as simplest fluctuations on the Fermi surface corresponding to an inverted harmonic oscillator potential [43]. Thus perhaps in the nonperturbative setup, the nontrivial time dependent deformations of this Euclidean Hartle Hawking vacuum in the (sub)critical theory is carrying out some kind of dimensional mutation by capturing the physics of a supercritical regime (see discussions in [51] and references therein for comments on dimensional mutation in closed string tachyon condensation in the context of cosmological singularities).

In this paper we put together all the ingredients of the problem in the context of the large N RG [1, 2], a Wilsonian world sheet RG, of Dirichlet boundaries in $2D$ noncritical string theory (the FZZT branes [52] with Dirichlet boundary condition on matter). They describe the quantum cosmology of the one dimensional universes with homogeneous matter. The wave function of the universe is given by the one point function of the loop operator $W(l,t) = \frac{1}{N}\mathrm{Tr}\exp[l\Phi(t)]$ [53, 54, 55, 56]. Note that the loop operators we study also have space-time interpretation. Here we compute the wave function by performing the large N RG of the one point function of the loop operator of the noncritical theory around a $c = 1$ fixed point at a string scale compactification radius (a self-dual radius $R \sim 1$). In fact, in our RG scheme, the Dirichlet boundary condition on matter at a $c = 1$ fixed point prefers a small compactification radius. As a result the large N RG deals with all the N^2 quantum mechanical degrees of freedom, instead of the N eigenvalues only as in minisuperspace free fermionic description. This translates into unavoidable large fluctuation of dilaton and nonlocal effects associated to the presence of the off diagonal elements (that can not be integrated out in $c = 1$ theory on a small circle) and presumably release of large number of nonperturbative states [57, 58, 59] causing a dimensional mutation. Thus the RG captures a nontrivially deformed condensate that modifies the WdW equation by scale derivatives of all orders, a nontrivial time dependence and nonlocal deformations corresponding to emission and absorption of point-like baby universes. Strikingly the process of emission and absorption of babies are controlled by $O(1/N)$. The short distance wave function of this modified WdW gives a tunneling wave function that corresponds to birth or nucleation of the universe into an inflationary de Sitter vacuum. Here let us briefly mention that in the recent semiclassical study of the decay of weakly metastable vacua in $2D$ Liouville gravity in [21], gravitational inflation inside the region of lower energy phase due to droplet fluctuation in the critical swelling process modifies the standard exponential suppression of the nucleation probability to a power law one, much like the behavior of the dynamical lattice Ising model. Now the far future wave function gives a global de Sitter as would have been given

[2] See [41, 48, 49] for the issue of particle production and thermality. Also see [50] for other time dependent solutions of similar w_∞ deformed condensates with a cosmological interpretation.

by a dS minisuperspace theory. The scale derivatives are playing an important role. Truncating them to quadratic power gives only the global de Sitter ($J_0(\sqrt{-\mu}l)$) in the far past like a de Sitter minisuperspace theory instead of a tunneling wave function. The arbitrary scale fluctuations help to fix the initial condition uniquely.

It is interesting to observe that, the time dependent nonperturbative deformations in non-critical theory is doing something nontrivial to enhance the effective space-time dimensions and hence accesses the Lorentzian de Sitter geometry dwelling in a supercritical regime. Presumably the dimensional mutation is due to large number of nonperturbative states belonging to the nonsinglet sector of the matrix quantum mechanics appearing at the self-dual radius. The real nature of these states (whether they are best described as world sheet vortices or long winding strings) is yet to be understood [57, 58, 59, 60] (also see [2, 61] for a discussion on the nature of these states in the context of $2D$ black hole). It has been pointed out in [51] that in the case of spacelike singularities studied in [28] and in similar cases with positively curved spatial slices, the singularity is lifted by a tachyon phase with fewer degrees of freedom, as opposed to the cases corresponding to negatively curved spaces. This is a candidate big bang case where with forward time evolution along the larger direction of the compactification circle the dimensionality (c_{eff}) increases. This is similar to the realistic inflationary cosmology where universe evolves in the direction of the evolution of de Sitter entropy [62, 63]). Note that, in our case the universe evolution is not in the direction of decreasing R, instead is in the direction of increasing l. The big bang singularity in our picture sits at small scale ($l \sim H^{-1}$) and is resolved by popping up of a simple tachyon condensate (Euclidean vacuum) at large R. Whereas tuning the radius to string scale deforms the condensate setting in dimensional mutation to supercritical regime. In this case an inflationary de Sitter vacuum pops up at $l \sim H^{-1}$ to resolve the singularity. Nevertheless the universe evolution in our picture does take place in the direction of evolving de Sitter entropy. This is because for $l < H^{-1}$ we have tunneling to the corresponding vacuum from nothing. Thus in effect the R direction maps to the set of initial condition in string theory that maps to the available set of vacuum (AdS or dS) through the large N RG (see the figure 1 below). However, as we will see, the dynamical vacuum selection here depends on other factors (dilaton fluctuation) that uniquely specify the initial tunneling wave function. As pointed out in [64], the universe we live in seems to have some mechanism of self tuning. Perhaps the large N RG incorporating dilaton fluctuation and nonlocal effects due to multiple world sheets inherently exhibits such self tuning that enables one to flow to a more realistic vacuum, starting around a $c = 1$ fixed point. It will be interesting to generalize the problem, in particular for an inhomogeneous universe with arbitrary scale and matter fluctuation, which we will report later [42].

The plan of the paper is as follows. In section 2 we review and elaborate the problem of inflationary de Sitter in $2D$ cosmology and summarize our results and critical observations.

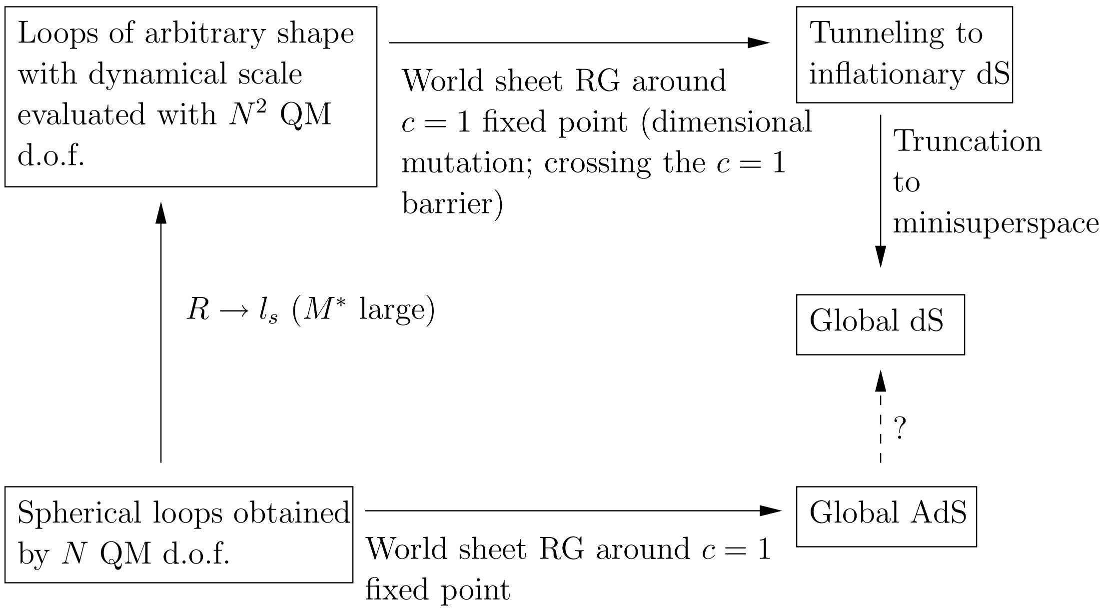

Figure 1: The mapping of the set of initial conditions in string theory to the set of available vacua. The broken arrow denotes possible time dependent deformations of the Fermi surface picture that may connect the two sets of vacua.

In section 3 we review our previous work [1] on large N RG analysis in $c = 1$ matrix model. In section 4 we deduce the familiar minisuperspace WdW equation of free fermions, namely the AdS_2 quantum mechanics, from large N RG. In section 5 we compute its deformation, the generalized WdW equation for one dimensional homogenous universe with arbitrary shape and scale fluctuations. In section 6 we identify the Hubble scale and then solve for the wave function at small and large scale limits and discuss the cosmological implications: tunneling to inflationary de Sitter in the far past and an expanding global de Sitter in the far future. In section 7 we discuss the topology changing amplitudes captured by large N RG and suppression of baby universes and self-tuning of cosmological constant.

2 $2D$ cosmology and inflationary de Sitter vacuum

Let us now review and elaborate on the basic philosophy of the problem in bosonic string theory.

2.1 The minisuperspace WdW cosmology in (sub)critical and super-critical theory

Let us consider Euclidean quantum gravity described by Liouville field theory coupled to $c = 1$ matter or its discrete matrix quantum mechanics version, where one has a good nonperturbative description of the time dependent physics. The tachyon field $T(\varphi, X(\sigma))$ on the world sheet is directly related to the macroscopic loop fields $W(l,t)$ [43, 44, 45, 53, 54, 55, 56] via

$$W(l,t) \sim e^{-\frac{1}{2}Q\varphi}\, T(\varphi, t)\,. \tag{2.1}$$

Here, φ is Liouville field and Q is the slope of the linear dilaton background ($Q = \sqrt{2}$ for $c = 1$ theory). The relation can be intuitively understood by considering the first order fluctuation δW that corresponds exactly to the tachyon wave function and satisfies minisuperspace WdW equation up to a factor of coupling constant. The vacuum expectation $\langle W(l,t)\rangle$ is given by the Hartle-Hawking type wave function $\psi(l)$. In the matrix quantum mechanics, these are nothing but the $FZZT$ branes or the one dimensional boundaries of length l

$$\psi(l) = \frac{1}{N}\langle \mathrm{Tr} e^{l\Phi(t)}\rangle\,, \tag{2.2}$$

with Dirichlet boundary condition on the world sheet matter

$$W(l,t)_{cont} \;\sim\; \delta(\oint e^{\frac{1}{2}\gamma\varphi} - l)\; \delta(X(\sigma) - t)\,. \tag{2.3}$$

Here $W(l,t)_{cont}$ represents the loop operator in the corresponding continuum Liouville theory. Their wave function, computed by the eigenvalue representation or the free fermion representation of matrix quantum mechanics [54, 55, 56], satisfies the same minisuperspace WdW equation as in the continuum Liouville field theory (accompanied by proper identification of the matrix model couplings with the continuum parameters)

$$\Big[-(l_0\frac{\partial}{\partial l_0})^2 + 4\mu l_0^2 + \nu^2\Big]\psi(l_0)_{h=0} = 0\,, \qquad l_0 = e^{\frac{\gamma}{2}\varphi_0}\,. \tag{2.4}$$

In matrix model computation, the all genus version of (2.4) for zero Fourier mode of matter modifies the WdW by adding a $-g_s^2 l_0^4$ term in the potential. Note that, the free fermion computation we mentioned here is computed in minisuperspace (*i.e.* φ space is truncated to φ_0). This is manifest in the massless scalar (the fluctuation of the eigenvalue density) quadratic action that exactly maps to the minisuperspace WdW action of the Liouville field in a suitable coordinate in the space of eigenvalues [43]. Both the continuum and the matrix model solutions for the minisuperspace Hartle-Hawking vacuum (an undeformed tachyon condensate) are just Bessel functions with simple FRW cosmology. The solution decaying at large length is the non-normalizable wave function $K_\nu(2\sqrt{-\mu}l_0)$. In fact $K_0(2\sqrt{-\mu}l_0)$ describes AdS vacuum ($\mu > 0$) that has simple big bang and big crunch [46]. Thus the free fermion quantum mechanics (2.4) with $\mu > 0$ is a quantum mechanics for AdS_2. From the point of view of loop operator, it describes dynamics of spherical loops with a homogeneous matter. Note that a wave function given by Bessel $J_0(2\sqrt{-\mu}l)$ and Hankel $H_0(2\sqrt{-\mu}l)$ would have described the behavior of global de Sitter in the far past and far future respectively [46], as it will arise in a different context. However, as we will see, these Bessel type of solutions describe merely homogeneous FRW geometries and do not seem to capture any inhomogeneity introduced by inflationary scenario with cosmological time evolution.

In terms of the free fermion phase space, the undeformed tachyon condensate can be described by the physics of a static Fermi sea corresponding to the eigenvalue hamiltonian with inverted oscillator potential filled up to a level μ. The tachyon wave function can be visualized as the fluctuations δW along the Fermi surface $H(p,\lambda)=\mu$. Now a general potential $V(\lambda)$ would spatially deform (fold, shift or open) the Fermi surface $H(p,\lambda)=p^2-\lambda^2=\mu$ to a Fermi surface of an arbitrary shape $H(p,\lambda)=p^2+\frac{1}{2}V(\lambda)=\mu$. This introduces higher derivatives in the minisuperspace WdW equation that modifies the genus zero tachyon wave function according to

$$\left[\partial_t^2+l_0^2V\left(\partial/\partial l_0\right)+\frac{l_0}{2}V'\left(\partial/\partial l_0\right)\right]\,\delta W_{h=0}=2\mu l_0^2\;\delta W_{h=0}\,. \tag{2.5}$$

Explicit time (matter) dependent deformation (tearing into droplet, opening and draining to the other side) of the Fermi surface result in nontrivial time dependent coefficients in the kinetic and potential terms of the WdW equation. Effectively it boils down to a time dependent $\mu(t)$ and a time dependent over all multiplication factor in the tachyon wave function. A class of integrable (w_∞) time dependent deformation have been shown to have cosmological interpretations (e.g. bounce cosmology) [50].

For inflationary scenario, one needs an appropriate temporal and spatial deformation of the condensate such that the modified WdW gives rise to some kind of nucleation to a de Sitter vacuum at UV (or in the far past) and its proliferation. Note that from (2.4), a naive continuation to the de Sitter geometry ($\mu<0$) by dialing the sign of cosmological constant μ is subtle and apparently does not make sense for the eigenvalue dynamics of the $c=1$ matter. There the $\mu\to 0$ corresponds to instability due to spilling of the eigenvalues from the top of the inverted oscillator potential (the continuum limit in the double scaling procedure) and the $\mu<0$ side is unbounded. However, considering an interesting class of w_∞ deformations corresponding to special class of time dependent perturbations of the Fermi sea (opening and draining of the Fermi sea to the other side), one can show formation of spacelike horizons either or both at past and future spacelike infinities [47] and also particle production [48, 41, 49]. Strikingly these properties indicate remnant of some kind of de Sitter quantum mechanics. Thus it is not meaningless in $c=1$ matrix model to look for most general (integrable and nonintegrable [3]) time dependent deformation of the tachyon condensate that will give rise to wave function exhibiting nucleation to de Sitter vacuum and possibly its proliferation too. For $c\geq 25$ matter a global

[3] All the integrable w_∞ deformations can be written in minisuperspace variable φ_0 whereas the most general deformations in (accessible) superspace are all not necessarily integrable.

de Sitter geometry at large scale can be probed by considering a Liouville-Sine-Gordon model of a inhomogeneous tachyon condensate [46]. The condensate is in the form of gravitationally dressed cosine matter potential in one of the matter (spatial) direction and is homogenous in remaining scalar fields that only play a role through the Weyl anomaly (they are all massless and remain in their initial state throughout the evolution of the universe). The minisuperspace WdW equation is then modified by a matter dependent cosine term in the potential. This gives a solution exhibiting global de Sitter geometry at large scale though it recollapses at small scale. However, such a condensate is still too simple to catch proliferation of the de Sitter or particle production.

In this respect one has to keep in mind that the quantum cosmology for $c = 1$ matter has few subtleties that are better dealt in $c \geq 25$ regime [36, 37, 35]. This is because the length scale of the universe behaves as $l = \oint d^2\sigma\, e^{\gamma\varphi}$, where $d^2\sigma e^{\gamma\varphi}$ is conformally invariant (*i.e.* $Q = \gamma + 2/\gamma$). For $c = 1$, Q and hence γ are real and the length scale grows proportionally with the string coupling $g_s \sim e^{\frac{1}{2}Q\varphi}$. The universe is then small at weak coupling and large at strong coupling, which is unrealistic for doing cosmology. For $c \geq 25$, Q and γ are purely imaginary. Thus one Wick rotates $\varphi \to -i\varphi$ getting a length scale inversely varying with string coupling $g_s \sim e^{-\frac{1}{2}Q\varphi}$. The wick rotation has an added advantage that it makes the kinetic energy of the time-like Liouville coordinate negative (analogous to that in $4d$ Euclidean gravity) so that the world sheet geometry has a Lorentzian interpretation. Thus for technical reasons we discussed so far, $c \geq 25$ matter has been preferred over the $c = 1$ matter theory in probing de Sitter cosmology on the world-sheet [46]. Finally for $c = 1$ theory at strong coupling, formation of baby universes of macroscopic size is favored that may absorb the inflating core of the universe and force it to recollapse, once the fluctuation drives the inflaton off its potential maxima. In single closed universe cosmology of gravity coupled to $c = 1$ matter in $\mathcal{R} \times S^1$, where the winding strings in a Sine-Gordon Liouville setup can provide topological defect acting as a seed, such a mechanism disallows possible eternal topological inflation.

2.2 The modified WdW cosmology: results and observations

In this paper, we use the general framework of large N RG approach in matrix quantum mechanics on a circle [1, 2] that works with N^2 quantum mechanical degrees of freedom instead of N eigenvalues and thus probes physics beyond the free fermionic approximation. The dilaton emerges in a more complicated way involving all the off-diagonal degrees of freedom and hence φ is not simply truncated to φ_0 but is dynamical in all the modes. Using this approach we derive a generalized WdW constraint and the corresponding (all genus) wave function for FZZT branes with Dirichlet boundary condition on matter. Thus the quantum cosmology we study is that of one dimensional closed universe of arbitrary shape $l = \oint d^2\sigma e^{\sqrt{2}\varphi}$ with homogeneous matter $t = \hat{x}$ and arbitrary scale fluctuation $\delta l \sim \oint d^2\sigma e^{\sqrt{2}\varphi}\sqrt{2}\delta\varphi$.

In matrix quantum mechanics picture the universe of length l is created by the macroscopic loop operator $\frac{1}{l}\mathrm{Tr}\phi_N^l(t)$ that cuts a hole of lattice length l on the discretized string world sheet. They have the same continuum limit as the lattice loop operator $W(l,t) = \frac{1}{N}\mathrm{Tr}\exp(l\ \phi_N(t)/a)$, where $\phi_N(t)$ are $N \times N$ hermitian matrices and a is a lattice spacing, and can be studied in this form. The commonly used boundary condition on these one-dimensional universes is the one that constrains the boundary length to be l throughout which the matter field takes a single value (say $t = \hat{x}$). It will be more useful to compute the wave function for the one point function of the loop operator with one puncture or the wave function for a boundary with one marked point

$$\Psi(l) = \frac{\psi(l)}{l}\,, \qquad \psi(l) = \langle W(l,\hat{x})\rangle = \langle N^{-1}\ \mathrm{Tr}\exp[l\phi(\hat{x})]\rangle\,. \tag{2.6}$$

A large length fluctuation over an arbitrary shape of loops is the dominant effect at small scale and hence is crucial to explain the inflationary scenario of early universe. Below we summarize our results and critical observations.

A nontrivial condensate

The world sheet RG modifies the WdW constraint by adding derivatives with respect to scale of all degrees and introducing nontrivial matter dependence. Along the line of our discussions above, this amounts to capturing physics of a nontrivial and complicated (perhaps nonintegrable) spatially deformed time dependent tachyon condensate. The inflaton field or (in this case) the matter field acquires a light (exponentially vanishing) mass and thus have no problem in executing a slow roll. The world sheet RG also adds a $O(1/N)$ correction that has inherently quantum mechanical origin coming from the anomaly in the Callan-Symanzik equation.

The quantum creation of inflationary de Sitter vacuum

The short distance $l < H^{-1} \sim M_p^{-1}$ wave function, which we obtain, is a tunneling wave function describing quantum creation of a closed universe from *nothing* and its nucleation to an inflationary de Sitter vacuum

$$\Psi(l < H^{-1}) \sim \exp[-cH^2 l^2]\,. \tag{2.7}$$

At large scale $l > H^{-1}$ the evolution of the universe is classical. Thus at large scale the wave function reproduces the semiclassical behavior of an expanding global de Sitter geometry, in spite of higher derivative and other quantum effects. The relevance of the higher derivatives at short distance can be realized from the fact that, in this domain a quadratic WdW will give rise to a far past global de Sitter ($J_0(\sqrt{-\mu}l)$, $\mu < 0$) only, instead of a tunneling wave function.

Suppression of baby universes

The baby universes generated (*i.e.* emitted or absorbed by loop splitting and reconnection) in this cosmology are of vanishing sizes and their formation is suppressed by higher orders in $1/N$. Their amplitude comes from nonlinear part of the WdW equation, generated by multi-trace contributions in the matrix quantum mechanics loop dynamics, that can be represented schematically by a complicated '$*$' operation

$$\mathcal{H}\psi + \psi * \psi = 0\,, \tag{2.8}$$

where $\mathcal{H}$ is the modified WdW operator as captured by the world sheet RG. The one point function ψ is a Hartle-Hawking wave function in the sense of [43]. The amplitude corresponding to the '$*$' operation is coming from some anomaly, breaking the scale invariance imposed by a linear WdW constraint $\mathcal{H}\psi = 0$, originating from nonlocal contribution from multiple world sheets. The vanishing size of the baby universes and suppression of their amplitude helps in continuation of inflation once it onsets. Note that, the birth of baby universes involves topology changing and thus inherently is a beyond minisuperspace process. Such amplitudes get contributions from the points in the interior of different topological sectors of the superspace [4]. We observe that the sub-leading contribution from the emission of a single baby universe self-tunes the de Sitter entropy to be small positive.

Emergence of Hubble scale

In $2D$ the gravitational coupling is dimensionless and there is no proper Planck scale. The Planck scale assumed to be spontaneously induced at $g_s \sim 1$. However, in the large N world sheet RG, the Planck scale emerges naturally as the Hubble scale for the problem. The potential in the generalized WdW equation for the one point function of loop operator with a marked point helps to identify the required inverse Hubble scale H^{-1}, which is very small for $c = 1$ case. This would be favorable for quantum tunneling of universe from nothing through an inflationary scenario. The potential below this scale has a tunneling type barrier. The geometry at H^{-1}, at which the universe pops out from the classically forbidden region, is a de Sitter geometry.

The open string moduli dependence and the geometry of the true vacuum

In our approach, the nonperturbative effects due to the N^2 quantum mechanical degrees of freedom is observed to be introducing explicit open string moduli dependence in the wave function[4] and in the Hubble scale. Such a dependence is not present in the wave function when we explicitly work with N quantum mechanical degrees of freedom. The open string moduli dependence does not directly affect the inflationary scenario, as the tunneling wave function solution exists for any sensible constant H. However, the explicit dependence of the Hubble constant on the open string moduli determines the geometry of the true vacuum one tunnels to. It will be extremely interesting to investigate whether the world sheet vortices are responsible for some kind of spontaneous symmetry breaking that chooses the ground state.

Choice of initial condition

A problem of describing creation of universe, in particular an inflationary universe, always has subtleties with choice of unique initial condition corresponding to 'nothing' [16]. This has long been a subject of big debate (see for example [5]). With introduction of more degrees of freedom as we go beyond minisuperspace and derivatives of all degrees, the situation is even worse. One would encounter an infinite dimensional initial condition (for example, specifying initial value of all the derivatives of the wave function) in all the degrees of freedom involved $(l_0, \delta l)$. However we observe that a gaussian profile of initial scale fluctuation $\psi(\delta l) \sim e^{-\delta l^2}$ provides all the required initial conditions and thus uniquely determines a tunneling wave function at UV. In other words, the wave function $\psi(\delta l)$ for initial scale fluctuation can be viewed as a co-moving (gravitational) wave detector to capture the inflationary vacuum.

The arbitrary scale fluctuations

The arbitrary scale fluctuations, taking us beyond the minisuperspace, are playing a pivotal role in defining a proper cosmological picture here. Because of the scale fluctuation, the actual volume of the one dimensional universe can no longer be definitely small at weak coupling and large at strong string coupling. A large fluctuation of the form $\delta\varphi \sim e^{-\sqrt{2}\varphi} - e^{+\sqrt{2}\varphi}$ would drive the scenario to the opposite. Now the length scale of the universe $l = \oint e^{\varphi_0 + e^{-\sqrt{2}\varphi} - e^{+\sqrt{2}\varphi}}$ becomes small as $\varphi \to \infty$ (strong coupling) and grows as $\varphi \to -\infty$ (weak coupling).

Amusingly, such a fluctuation in φ would imply a scale dependence of the power spectrum of primordial density fluctuation for our one dimensional universe (naively, the density perturbation $\frac{\delta\rho}{\rho} \sim -\delta\phi \sim e^{+\sqrt{2}\varphi} \sim l$). Strikingly, the tunneling wave function fixed by the initial gaussian scale fluctuation profile captures a vanishingly small power-like scale dependence (like a real-world nearly non flat CMB anisotropy). Let us define the probability of finding the universe between a scale l and $l + \delta l$ as $l \gtrsim H^{-1}$ at a time 0+ after the big bang to be $\frac{\delta l}{l} = P(l \to l + \delta l) = \frac{\psi(l+\delta l)}{\psi(l)} \sim \left(\frac{H^{-1}}{l}\right)^2$. For a small universe ($l < 1$) and for an estimate

[4]We thank Micha Berkooz for a discussion on this issue.

$l \sim H^{-0.99}$ to realize the bound $l \gtrsim H^{-1}$, the power spectrum is slightly red-shifted

$$\frac{\delta\rho}{\rho} \sim l^{0.02}\,, \quad n_s \sim 0.96\,. \tag{2.9}$$

Arbitrary matter fluctuations

An arbitrary matter fluctuation is crucial to capture the particle production and also the formation of large scale inhomogeneities in inflation (as seen by comoving particle detectors). In matrix quantum mechanics this is a more delicate issue as it would require to consider wave function for a $D1$ brane with Neumann boundary condition on both Liouville and matter coordinates. In some sense this implies dominance of the kinetic term in the matrix quantum mechanics action (hence presumably the dominance of the nonsinglet sector), over the potential term, that transforms as singlet under $SU(N)$ rotation. The angular (off-diagonal) degrees of freedom in the kinetic term can not be integrated out except for large compactification radius or for matrix quantum mechanics on a line [57, 58, 59]. In other words, it would need a full open string description of matrix quantum mechanics. In [67] the problem have been dealt to some extent by considering interacting spin chains. In an upcoming paper [42] we will extend the case studied here to an inhomogeneous universe (the Neumann boundary condition on matter) to study arbitrary matter fluctuations. The case with Neumann boundary conditions in both matter and Liouville directions has not yet been solved in matrix quantum mechanics.

The factor ordering ambiguity

The arbitrary scale fluctuation allows a small universe at strong coupling. As $l \to 0$, the potential in the WdW equation $V(l, \hat{x}) \to 0$ and hence the initial wave function could be incoming or outgoing plane waves (FRW geometries). However, large N RG captures a factor ordering ambiguity $p \neq 1$ of the quadratic differential operator $(l^p \frac{\partial}{\partial l} l^p \frac{\partial}{\partial l})$ in the WdW equation. This bars the wave function from being only plane wave at small scale, thus encouraging an inflationary scenario.

Instability and third quantization

Intuitively one can imagine for eternal inflation, in which the space-time of universe nucleates in a fractal structure, a violent fluctuation in all modes of φ would be necessary. On the other hand, such extra dynamical fields beyond minisuperspace usually give rise to unstable scalars and hence negative norm states leading to recollapse of the universe [5]. However, in our approach, the situation is quite different. The theory of loops on world sheet can be seen as some kind of third quantization (see for example [43, 55] and references therein). Then the problem of negative norm state is not relevant as the wave function of the universe is actually a quantum field operator (the loop operator itself) composed of creation and annihilation operators for the universe and the WdW equation follows from the third quantized action. The generalized WdW operator in large N RG emerges as a natural gauge condition for loops of arbitrary shape with arbitrary scale fluctuations (from the anomaly in the Callan-Symanzik equation for the loop operator), that takes care of negative norm states of the dynamical fields.

The minisuperspace limit

To discuss the validity of our derivations we reproduce the minisuperspace AdS_2 quantum mechanics by large N RG on the space of matrix eigenvalues. It is very interesting to note that the world sheet RG somehow captures a self-tuned deformation of the AdS_2 quantum mechanics of the N degrees of freedom (the matrix eigenvalues) to that of a dS_2 quantum mechanics of all the N^2 degrees of freedom. Perhaps it is an indication that there is such a self-tuning mechanism of dynamical vacuum selection in real world that we will be able to capture by going beyond the minisuperspace approximation.

D-instantonic picture for tunneling?

It will be certainly interesting to understand if there is any instantonic picture for the tunneling. In free fermionic representation, the vacuum with de Sitter sign of the cosmological constant ($\mu < 0$) has an instability due to an instantonic tunneling amplitude $O(e^{-N})$ from the other (AdS) side and is central to the nonperturbative formulation of the theory [68]. The Liouville theory coupled to $c = 1$ matter does have D-instantons (ZZ branes) of $c = 1$ noncritical string theory localized in the strong coupling region ($\varphi \to 0$) [69]. It has been interpreted that the FZZT brane (extended in $\varphi \to -\infty$) probes the φ direction up to a certain scale (until $g_s \sim 1$ analogous to $l \sim H^{-1}$ in our cosmology), beyond which it dissolves. Below this scale the open string degrees of freedom are only these D-instantons stuck at $\varphi \to \infty$ which can be thought of as the remnant from the superposition of a FZZT brane stretching from weak to strong coupling and then back to weak coupling region [70]. Thus the wave function for FZZT branes at small scale $l < H^{-1}$ actually computes a D-instanton tunneling amplitude in all genus. Note that, unlike the tunneling event in N degrees of freedom picture (eigenvalue representation), here the tunneling event captured by the dynamics of N^2 degrees of freedom ends in a stable de Sitter vacuum.

An artifact of the formalism?

One might wonder whether all these nice aspects captured by large N RG is in some sense an artifact of the formalism itself or they are really happening. To this end, we need to clarify whether the small matrix field (small $\Phi_N(t)$) approximation, which is the only approximation in the large N RG, is in contradiction with the large eigenvalue at the strong coupling (eigenvalue D-instanton) or at the point of spilling [5], or more generally to a large quantum fluctuation in the matrix field variable itself. However, in integrating out each momentum shell $(v_{1\times N}\,, v^*_{N\times 1}\,, \lambda_{N+1} = 0)$ from a $(N+1)\times(N+1)$ matrix $\Phi_{N+1}(t)$, the remaining $\Phi_N(t)$ part that enters in the interaction vertex $g\ \Phi_N$ is never approximated to be small in an absolute sense. Rather it is assumed lighter compared to the integrated out degrees of freedom which become heavier by a rescaling $v \to \tilde{v} = \sqrt{N}v$. This makes the interaction vertex lighter compared to external integrated out momenta, thus facilitating in the expansion of its exponential to get Feynman diagrams.

World sheet vs space-time observer?

Our study of the cosmology is performed with respect to the world sheet observer, which is meaningful as the observer should be a part of the universe. However, even though the wave function for the universe we are describing is computed on the world sheet, the loop operators do have a proper space time interpretation and the cosmology could be translated on that language.

Euclidean vs Lorentzian picture?

It appears that the world sheet large N RG with N^2 quantum mechanical degrees of freedom automatically captures the Lorentzian de Sitter vacuum of the supercritical theory instead of the Euclidean AdS vacuum of the subcritical theory, the later being captured by the large N RG of N quantum mechanical degrees of freedom. Perhaps the world sheet RG has some nontrivial dynamics that leads to a growth of space-time dimensions[6].

In figure 2 below, we summarize the main results in a schematic form. The broken arrow represents a possible w_∞ deformations showing past and future spacelike infinities as was also included in figure 1.

3 The large N RG in MQM

We will now review in some detail the steps of the large N world sheet RG analysis in MQM. For a complete description see the original works in [1, 2]. The concept arises from the interpretation of the very existence of the *double scaling limit* (or the continuum limit) as some kind of Wilsonian RG flow [65].

[5]We thank Emil Martinec for pointing this out.

[6]We thank Eva Silverstein for a discussion on this.

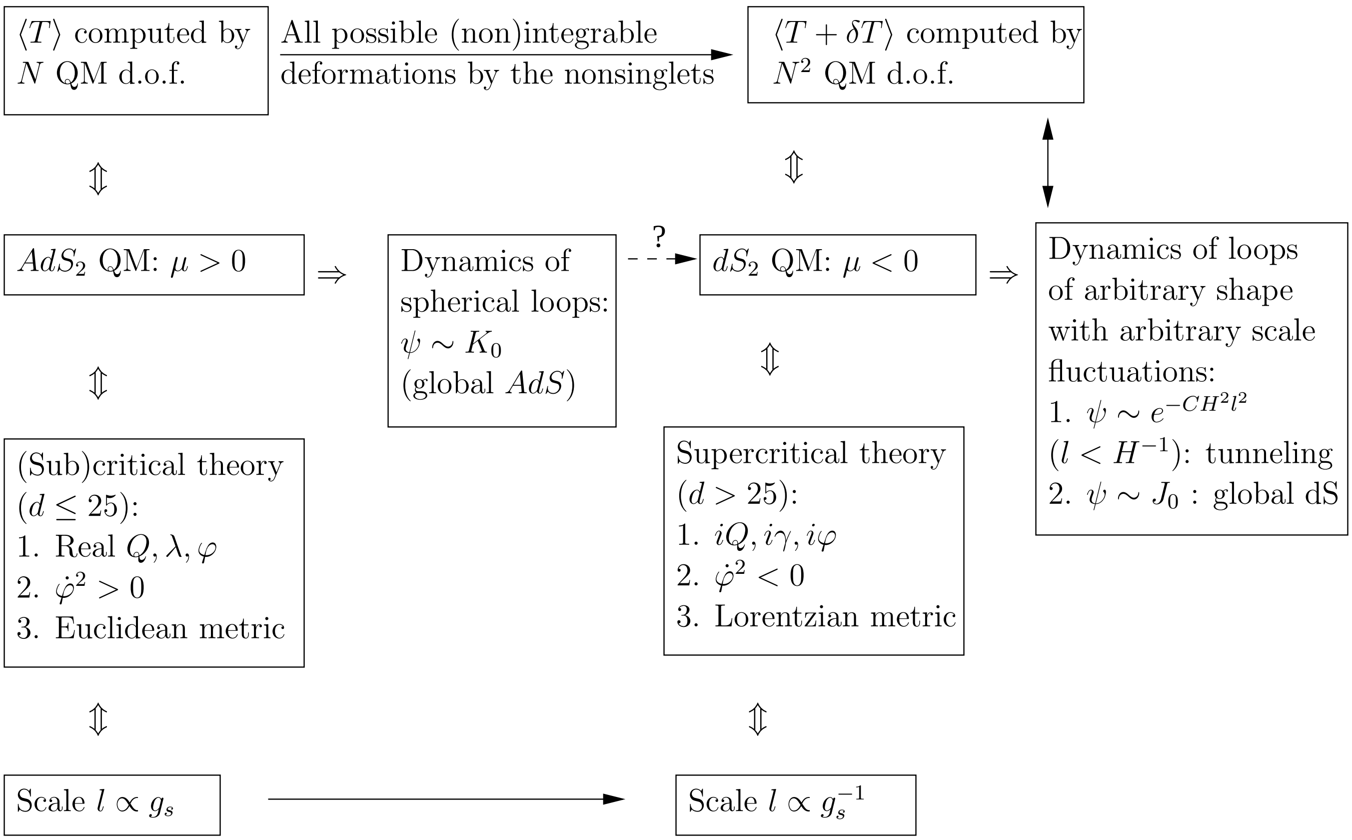

Figure 2: Summary of results and observations.

In the double scaling limit, as the matrix coupling constant $g \to g_c$, the average number of triangles in triangulations at any genus G diverges as

$$\langle n_G \rangle \sim (1-G)(\gamma_0 - 2)(1 - g/g_c)^{-1}\,. \tag{3.1}$$

Simultaneously with $N \to \infty$, the length of the triangles (the regularized spacing of the random lattice) $a \sim N^{-\frac{1}{2-\gamma_0}}$ scales to zero to keep the physical area $a^2\langle n_G\rangle \sim N^{-\frac{2}{2-\gamma_0}}(1-g/g_c)$ or equivalently the string coupling $g_s \equiv N^2(g-g_c)^{2-\gamma_0}$ fixed. The existence of double scaling limit indicates that a change in the length scale induces flow in the coupling constants of the theory in a way that one reaches the continuum limit with correct scaling laws and the critical exponents at the nontrivial IR fixed point determined by the flow equations. The large N world sheet RG in MQM [1, 2] is basically the evolution of the two sets of parameters of the theory, the size of the matrices N and the cosmological constant (mapped into the matrix coupling g) and all other matrix couplings, at the constant long distance physics with the rescaling of the regularization length in the triangulation of the world-sheet. In the Wilsonian sense this is done by changing $N \to N + \delta N$ by integrating out some of the matrix elements, which is like integrating over the momentum shell $\Lambda - d\Lambda < |p| < \Lambda$, and compensating it by enlarging the space of the coupling constants $g \to g + \delta g$. Here the space of coupling constants will contain both the matrix coupling g and the mass parameter M^2.

3.1 The RG scheme

Let us now schematically explain the emergence of the flow equations. By integrating out a column and a row of an $(N+1) \times (N+1)$ matrix, the $c = 1$ matrix partition function satisfies a discrete relation

$$Z_{N+1}(g, M, R) = [\lambda(g, M, R)]^{N^2} Z_N(g', M', R') \tag{3.2}$$

with the following flow equations for the couplings

$$\begin{aligned} g' &= g + \frac{1}{N}\beta_g(g, M, R) + O\Big(\frac{1}{N^2}\Big)\,, \\ M'^2 &= M^2 + \frac{1}{N}\beta_{M^2}(g, M, R) + O\Big(\frac{1}{N^2}\Big)\,, \end{aligned} \tag{3.3}$$

and an auxiliary flow equation

$$\lambda(g, M, R) = 1 + N^{-1}\, r(g, M, R) + O(N^{-2})\,. \tag{3.4}$$

Here g', M' represent the renormalized couplings. Identifying $1/N$ as the world sheet scale of the RG transformation, the functions β_g, β_M are interpreted to be the beta functions for the corresponding couplings. The function λ gives the change in the world sheet free energy (the string partition function)

$$\mathcal{F}(N, g, M, R) = \frac{1}{N^2} \ln Z_N(g, M, R)\,. \tag{3.5}$$

The auxiliary flow relation for β_λ is useful in studying thermodynamical consequences. As N is taken to be large, the world sheet scale $1/N$ becomes infinitesimal. Then the discrete relation (3.2) for the Matrix partition function actually translates into a continuous Callan-Symanzik equation satisfied by the world sheet free energy with the function r controlling it's inhomogeneous part

$$\Big[N\frac{\partial}{\partial N} - \beta(g, M, R)\frac{\partial}{\partial g} - \beta(g, M, R)\frac{\partial}{\partial M} + \gamma\Big]\mathcal{F}(N, g, M, R) = r(g, M, R)\,. \tag{3.6}$$

Here γ acts as the anomalous dimension. We will see that the β_g, β_M computed by the large N RG are such that the homogenous part of the Callan-Symanzik equation indeed determines the correct scaling exponents for $c = 1$ matrix model around the nontrivial fixed point. The inhomogeneous part is related to some subtleties in the theory, like the *logarithmic scaling violation* of the $c = 1$ matrix model.

3.2 Integrating over the vectors (bosonic quarks)

To carry out the explicit derivation of the flow equations, let us consider the usual hermitian matrix quantum mechanics for $(N+1)\times(N+1)$ matrices $\phi_{N+1}(t)$ with a periodic boundary condition on the matrices. It has a cubic potential acting as a wall stabilizing the inverted oscillator potential. Now considering the following parametrization (the scalar being of relative order $1/N$ is neglected in the double scaling limit),

$$\phi_{N+1}(t)=\begin{pmatrix}\phi_N(t) & v_a(t)\\ v_a^*(t) & 0\end{pmatrix}, \tag{3.7}$$

and considering the power expansion (the higher order terms in v^*v are suppressed by powers of $O(1/N)$),

$$\begin{aligned}\mathrm{Tr}\phi_{N+1}^{2k} &= \mathrm{Tr}\phi_N^k + 2k\ v^*\phi_N^{2k-2}v + O(v^4)\,,\\ \mathrm{Tr}\phi_{N+1}^{2k+1} &= \mathrm{Tr}\phi_N^k + (2k+1)\ v^*\phi_N^{2k-1}v + O(v^4\phi^{2k-3})\,,\end{aligned} \tag{3.8}$$

the matrix partition function reduces to a part containing $N\times N$ matrices and an integral over the vectors

$$\begin{aligned}\mathcal{Z}_{N+1}[g,M,R] \;=\; & \int_{\phi_N(2\pi R)=\phi_N(0)} \mathcal{D}^{N^2}\phi_N(t)\\ & \exp[-(N+1)\mathrm{Tr}\int_0^{2\pi R} dt\ \{\frac{1}{2}\dot{\phi}_N^2(t)+\frac{1}{2}M^2\phi_N^2(t)-\frac{g}{3}\phi_N^3(t)\}]\\ & \int_{v,v^*(2\pi R)=v,v^*(0)} \mathcal{D}^N v(t)\mathcal{D}^N v^*(t)\\ & \exp[-(N+1)\int_0^{2\pi R} dt\ v^*(t)\{-\partial_t^2+M^2-g\phi_N(t)\}v(t)]\,.\end{aligned} \tag{3.9}$$

This is nothing but the $c=1$ partition function suitable for Veneziano type QCD considered in [66, 67] to study open strings. Both color N and fermion (quark) flavor N_f are taken to be large. Here the quarks are bosonic (the vectors) and $N=N_f$. It is precisely these fields in the fundamental representation of the global $SU(N)$ group, which generate boundary terms in the Feynman diagrams. Integrating adiabatically over these (bosonic) quark loops generates a logarithmic interaction term $(+N\ \mathrm{Tr}\log[-\partial_t^2+M^2-g\phi_N(t)])$ and hence the determinant $\det||-\partial_t^2+M^2-g\phi_N||^{-1}$ that gives rise to the boundaries in the world-sheet. Logarithm with minus sign would arise if the integration is performed on N flavors of fermions. Unlike the open string models mentioned above, here the couplings in front of the logarithm and in its argument are nor introduced by hand, rather they are determined by the original closed string action. One can ignore the derivative term inside the logarithm if the mass and the couplings are large enough. This makes the dynamical loops uncorrelated in time and gives rise to Dirichlet boundaries on the world sheet. On the other hand, if the derivative term dominates then the world sheet boundaries are Neumann. In [67], $c = 1$ model with explicit expression for the fundamental fields has been considered and is the one which is closest to our model obtained after integrating out one row and one column of flavor degrees of freedom. If one considers infinite line, only ground state is relevant. For some choices of the coupling and the mass of the particle moving around the boundary of the holes, it has been possible to find the ground state [67], but the exact spectrum is not known. Our ultimate goal is to derive the exact spectrum of these boundaries and to study their cosmological implications.

3.3 Evaluating $\det|| - \partial_t^2 + M^2 - g\phi_N||^{-1}$

Let us now evaluate the determinant by standard Feynman expansion. In general it is a nonlocal object. For simplicity one could consider the constant ϕ_N mode, which is equivalent to studying the effective potential to determine the phase structure. But here we will treat more general case of evaluating the determinant with flavors coupling to general ϕ_N. The complicated induced interactions arising from the logarithmic term are ignored in small field approximation in order to get back the beta functions that depend only on the original couplings. In terms of the Fourier modes, the v-integration can be expressed as

$$I[g, M, \phi_N, R] = \frac{1}{[\sqrt{2\pi R(N+1)}]^{2N}} \int \big(\prod_n dv_n^* dv_n^*\big) \; \exp[-\sum_m v_m^* \big(\frac{m^2}{R^2} + M^2\big) v_m + g \sum_{m,l} v_m^* \phi_{m-l} v_l] \,. \tag{3.10}$$

The integral can be re-casted in terms of a gaussian part I_0 being acted upon by a part generating interaction vertices

$$I[g, M, \phi_N, R, N] = \frac{1}{[2\pi R(N+1)]^N} \; \exp\Big[-\sum_{m_1,l_1} \mathcal{O}^{v^*v}_{m_1-l_1}(g,\phi)\Big] \; I_0(R, M, N)\,, \tag{3.11}$$

with the gaussian part defined as

$$I_0[R, M, N] = \int \Big(\prod_j dv_j^* dv_j\Big) \; \exp\big[-\sum_m v_m^* \mathcal{O}^{v^*v}_{mm} v_m\big]\,. \tag{3.12}$$

Here the operators $\mathcal{O}^{v^*v}_{mn}$ and $\mathcal{O}^{v^*v}_{m-l}(g,\phi)$ respectively define the inverse of propagator carrying a momentum $mn/R\delta_{mn}$ and interaction vertex carrying a net momentum $(m-l)/R$

$$\mathcal{O}^{v^*v}_{mn} = \Big(\frac{mn}{R^2} + M^2\Big)\delta_{mn}\,, \quad \mathcal{O}^{v^*v}_{m-l}(g,\phi) = g\phi_{m-l}\,. \tag{3.13}$$

The inverse of these operators define various propagators and vertices according to figure 3.

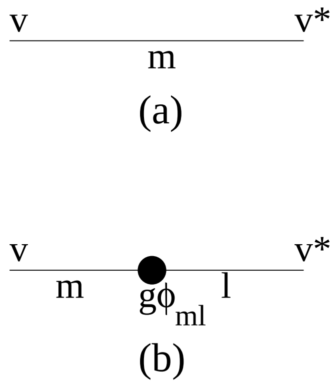

Figure 3: The propagators and vertices: (a) $[\mathcal{O}^{v^*v}_{mn}]^{-1}$, (b) $[\mathcal{O}^{v^*v}_{m-l}(g,\phi)]^{-1}$.

The gaussian integration is basically absorbed in a net prefactor in (3.11) such that it reduces to the sum of one loop Feynman diagrams $\Sigma[g, M, \phi_N, R, N]$ up to this prefactor. The sum $\Sigma[g, M, \phi_N, R, N]$ is given by

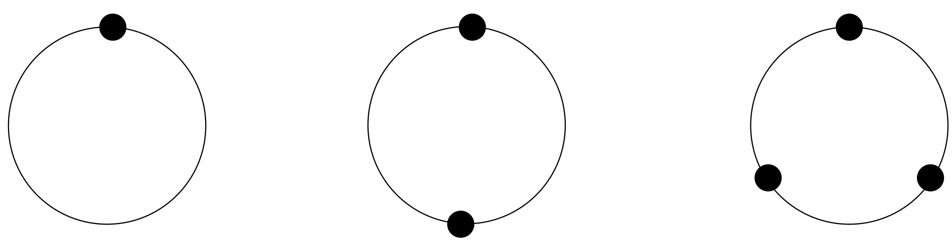

Figure 4: The diagrams contributing to Σ at one-loop order.

$$\begin{aligned}
\Sigma[g,\phi_N,R,N] &= 1 + g\ \mathrm{Tr}\Bigg[\sum_n \frac{1}{\frac{n^2}{R^2}+M^2}\ \phi_0\Bigg] \\
&+\frac{g^2}{2}\ \mathrm{Tr}\Bigg[\sum_{m,n}\frac{1}{\big(\frac{m^2}{R^2}+M^2\big)\big(\frac{n^2}{R^2}+M^2\big)}\ \phi_{m-n}\phi_{n-m}\Bigg] \\
&+\frac{g^3}{3!}\ \mathrm{Tr}\Bigg[\sum_{m,n,l}\frac{1}{\big(\frac{m^2}{R^2}+M^2\big)\big(\frac{n^2}{R^2}+M^2\big)\big(\frac{l^2}{R^2}+M^2\big)}\ \phi_{m-l}\phi_{l-n}\phi_{n-m}\Bigg] + O(g^4)
\end{aligned} \tag{3.14}$$

Now the one loop correction to the various coefficients in the effective action is evaluated term by term by inverse transforming the Fourier modes, summing up the set of infinite series and then exponentiating and log-expanding $\Sigma[g, M, \phi_N, R, N]$ using small field approximation. The summation is done according to

$$\sum_{m=-\infty}^{\infty} \frac{\exp[i(m/R)t]}{\frac{m^2}{R^2} + M^2} = \frac{\pi R}{M} \frac{\cosh(\pi MR - Mt)}{\sinh \pi MR}\,, \quad 0 \leq t \leq 2\pi R\,. \tag{3.15}$$

Since after performing the inverse Fourier transform, the various terms has nonlocal integrals over several one dimensional dummy time variables, we breakup the variables into center of mass and relative coordinates. Then we expand the functions about the center of mass coordinates, assuming the relative coordinates to be small enough, and consider integration over the relative coordinates. The higher derivative terms are dropped due to smallness of the relative coordinate compared to the center of mass coordinate. For example, by the coordinate transformation $(t_1, t_2) \to (T = (t_1 + t_2)/2, \tau = (t_1 - t_2)/2)$, the $O(g^2)$ term of the form $\int dt_1 \int dt_2\ \mathrm{Tr}[\phi(t_1)\phi(t_2)]$ can be expanded around T, considering τ small (which is true for $t_2 > 0$). Then the $\dot{\phi}^2$ term comes from $O(\tau^2)$ term in the expansion, whereas the next higher derivative term appears at $O(\tau^4)$. All the mixed terms of powers of ϕ and the powers of its derivatives appear in odd orders of τ and hence are automatically dropped due to integration of τ over a symmetric range.

Collecting all the contributions up to $O(\phi\phi\phi)$, the expression for $\Sigma[g, M, \phi_N, R, N]$ becomes

$$\begin{aligned} \Sigma[g, M, \phi_N, R, N] \ \simeq\ & 1 + F_{g1}(R, M)\ g \int_0^{2\pi R} dt\ \mathrm{Tr}\phi_N(t) + F_{g2}(R, M)\ g^2 \int_0^{2\pi R} dt\ \mathrm{Tr}\phi_N^2(t) \\ & + \hat{F}_{g2}(R, M)\ g^2 \int_0^{2\pi R} dt\ \frac{1}{2}\mathrm{Tr}\dot{\phi}_N^2(t) + F_{g3}(R, M)\ g^3 \int_0^{2\pi R} dt\ \frac{1}{3}\mathrm{Tr}\phi_N^3(t)\,. \end{aligned} \tag{3.16}$$

Note that, in the evaluation of Σ we keep the contribution from the nonlocal terms of the action up to the kinetic term and the contribution from the higher order terms in the matrix field up to the cubic term. All other operators are redundant for our purpose and are negligible due to the small field approximation. The coefficients $F(R)$s are defined as follows

$$
\begin{aligned}
F_{g1}(R,M) \quad &= \frac{1}{2M}\coth\pi MR\,, \\
F_{g2}(R,M) \quad &= \frac{1}{M^3\sinh^2\pi MR}\Big(\frac{1}{2}\pi MR\cosh 2\pi MR + \frac{1}{8}\sinh 4\pi MR\Big)\,, \\
\hat{F}_{g2}(R,M) \quad &= \frac{1}{M^5\sinh^2\pi MR}\Big(\frac{\pi MR}{16}\cosh 4\pi MR - \frac{\pi^3M^3R^3}{6}\cosh 2\pi MR \\
&\quad -\frac{1}{64}(1+8\pi^2M^2R^2)\sinh 4\pi MR\Big)\,, \\
F_{g3}(R,M) \quad &= \frac{\pi MR}{64M^5\sinh^3\pi MR(\cosh 2\pi MR + \cosh 4\pi MR)} \\
&\quad [4\pi MR\big(3\cosh\pi MR + 2\cosh 3\pi MR + 2\cosh 5\pi MR + \cosh 7\pi MR\big) \\
&\quad + \sinh\pi MR + \sinh 3\pi MR + \sinh 5\pi MR + 2\sinh 7\pi MR + \sinh 9\pi MR]\,.
\end{aligned}
\tag{3.17}
$$

The tadpole term linear in $\phi_N(t)$ is now removed by shifting the background and setting it's net coefficient to zero. This fixes the value of the shift ($f \sim -gM^2F_{g1}/N + O(\frac{1}{N^2})$) and accordingly modifies the coefficients of all the terms in the action.

3.4 Restoring the original cut-off and the space-time geometry around a fixed point

As usual in Wilsonian RG we now restore the original cut-off by rescaling the variables

$$
\phi_N(t) \to \rho\phi'_N(t')\,, \quad t' \to t(1-h\;dL)\,, \quad R' \to R(1-h\;dL)\,. \tag{3.18}
$$

Here $dL = 1/N$ and h is a function of the radius and the couplings whose functional form can be determined from the behavior of the flow equations near the fixed points. As one approaches a fixed point, h saturates to a constant characteristic to that particular fixed point, in the sense that it determines the scaling exponents. Thus the rescaling of the radius R in some sense tells us that as the system flows to various fixed points, the target space geometry changes accordingly. This is something new in matrix model which directly enables us to determine the target space metric around a fixed point by solving the equation in it's neighborhood

$$
\frac{dR}{dL} = -hR\,. \tag{3.19}
$$

For example, later in this section we will see that $h = 0$ corresponds to a $c = 1$ fixed point. This trivial rescaling indeed gives a flat metric ($R^2 = const$). In [72], this principle is used to extract $2D$ Euclidean black hole metric from matrix quantum mechanics.

3.5 Flow equations and the nontrivial fixed point

We now proceed towards determining the flow equations by setting the overall coefficient of the kinetic term to one. This gives ρ as

$$\rho = 1 + \frac{1}{2}(h - 1 + g^2 \hat{F}_{g2}) dL + O(dL^2)\,. \tag{3.20}$$

The resulting effective action is of the same form as the bare one with the renormalized strength of the couplings. The renormalized partition function is given by

$$\begin{aligned}
Z_{N+1} &= \lambda'^{\,N^2} \int \mathcal{D}^{N^2}\phi'_N(t')\ \exp\Big[- N\ \mathrm{Tr}\int_0^{2\pi R'} dt' \Big(\frac{\dot{\phi}_N'^{\,2}(t')}{2} + M'^2 \frac{\phi_N'^{\,2}(t')}{2} - g' \frac{\phi_N'^{\,3}(t')}{3} \Big)\Big]\,, \\
g' &= g + \Big(\frac{5}{2}h - \frac{1}{2}\Big) g\ dl + \big[F_{g3}(R,M) - \frac{3}{2}\hat{F}_{g2}(R,M)\big] g^3\ dL\,, \\
M'^2 &= M^2 + \big[2hM^2 + g^2(1-M^2)F_{g2}(R,M) - g^2 M^2 \hat{F}_{g2}(R,M)\big] dL\,.
\end{aligned} \tag{3.21}$$

The resulting beta function equations are

$$\begin{aligned}
\beta_g &= \frac{dg}{dL} = \Big(\frac{5}{2}h - \frac{1}{2}\Big) g + \big[F_{g3}(R,M) - \frac{3}{2}\hat{F}_{g2}(R,M)\big] g^3\,, \\
\beta_{M^2} &= \frac{dM^2}{dL} = 2hM^2 + g^2[(1-M^2)F_{g2}(R,M) - M^2 \hat{F}_{g2}(R,M)]\,.
\end{aligned} \tag{3.22}$$

As we already mentioned much of the structure of the beta functions depend on understanding the quantity h. The gaussian fixed point $\Lambda_1^* = \big(0,0\big)$ satisfies both the equations $\beta_g = 0$ and $\beta_{M^2} = 0$ trivially for any h. One can show that for $h = 0$, or in other words for trivial rescaling ($t' = t$, $R' = R$) of the target space coordinates, the nontrivial fixed points $\Lambda_2^*(g^{*2} \neq 0, M^* \neq 0)$ have $c = 1$ critical exponents. Such a pair of nontrivial $c = 1$ fixed points of the flow equations (3.22) exists for any R and are given by

$$\begin{aligned}
g^{*2} &= \frac{1}{2F_{g3}(R,M^*) - 3\hat{F}_{g2}(R,M^*)}\,, \\
&(1 - M^{*2})F_{g2}(R,M^*) - M^{*2}\hat{F}_{g2}(R,M^*) = 0\,.
\end{aligned} \tag{3.23}$$

The second equation solves for the values of M^* for different R. The fixed point values of the couplings are scheme dependent. As such their explicit values do not affect the physics as long as they are finite. However, to study the Dirichlet boundaries on the world sheet, we are interested in large values of M^*. As one cuts a hole on the triangulated world sheet, there is no longer any matter or the matrix valued fields sitting inside the cut portion. Instead there are vectors v, v^* who sew the cut portion by correlating the matter at the different points on the boundary. As their propagator goes like $1/M^*$, a large M^* would imply uncorrelated matter on the boundary and hence will serve as a Dirichlet boundary condition [67]. For $h = 0$, solving the corresponding equation for M^* (3.23), we find that a large M^* is feasible for $\pi R \leq 1$. This implies that the effects of nonsinglet sector cannot be suppressed in this situation. In fact M^* can have any value at vanishing R and is of order one in the range $0 < \pi R < 1$. Around $\pi R \sim 1$ it goes to a peak ($M^* \sim 2$) and then rapidly goes to zero.

3.6 The $c = 1$ critical exponent

Let us now explain how a nontrivial fixed point of (3.22) with $h = 0$ corresponds to a $c = 1$ fixed point. The critical exponents for the scaling variables, the renormalized bulk cosmological constant $\Delta = 1 - g/g^*$ and the renormalized mass (related to the renormalized boundary cosmological constant) $\mathcal{M} = 1 - M/M^*$ can be determined from the eigenvalues of the matrix

$$\Omega_{k,l} = \frac{\partial \beta_k(\Lambda^*)}{\partial \Lambda_l} \,. \tag{3.24}$$

The homogeneous part of the Callan-Symanzik equation, satisfied by the singular part of the free energy, can now be rewritten as

$$\Big[N \frac{\partial}{\partial N} - \Omega_1 \, \Delta \frac{\partial}{\partial \Delta} - \Omega_2 \, \mathcal{M} \frac{\partial}{\partial \, \mathcal{M}} + 2 \Big] \mathcal{F}_s \; [\, \Delta, \mathcal{M}, R] = 0 \,. \tag{3.25}$$

The eigenvalues Ω_is of the matrix $\Omega_{k,l}$ are nothing but the scaling dimensions of the relevant operators. The singular behavior with respect to the renormalized bulk cosmological constant goes as,

$$\mathcal{F}_s \sim \Delta^{2/\Omega_1} f_1[N^{\Omega_1} \, \Delta \,] \; f_2[N^{\Omega_2} \, \mathcal{M} \,] \,. \tag{3.26}$$

Comparing the above expression of $\mathcal{F}_s$ with the matrix model result $\mathcal{F}_s \sim \Delta^{(2-\gamma_0)} \; f[N^{2/\gamma_1} \Delta]$, or using the standard definition of the susceptibility $\Gamma \sim \frac{\partial^2 \mathcal{F}_s}{\partial \Delta^2}|_{\mathcal{M}=0} \sim \Delta^{-\gamma_0}$, the string susceptibility exponent γ_0 is given by

$$\gamma_0 \sim (2 - 2/\Omega_1) \,. \tag{3.27}$$

Note that in our analysis $\gamma_1 \sim 2/\Omega_1$ is consistent with the matrix model relation $\gamma_0 + \gamma_1 = 2$. This relation is independent of the explicit values of γ_0 and γ_1 and is easily obtainable from the consideration of the torus. The string susceptibility exponent at genus G is defined by

$$\gamma_G = \gamma_0 + G \; \gamma_1 \,. \tag{3.28}$$

For the nontrivial fixed point, $\Omega_{11} = 1 + 5h/2$. Note that in this RG, typically g^* is small. The pair of nontrivial fixed points is always situated close to the Gaussian fixed point. This implies the off-diagonal elements $\Omega_{12},\ \Omega_{21}$, being proportional to the powers of g^*, are small and $\Omega_{22} \sim h$. The scaling dimension matrix is effectively a diagonal one with $\Omega_1 = 1+5h/2,\ \Omega_2 = h$. Thus $h = 0$ gives the $c = 1$ critical exponent, $\gamma_0 = 0$. This is true for any R. The pair of $c = 1$ fixed points we thus get are repulsive with respect to the flow of the parameter g while the gaussian fixed point is attractive. One can show that this pair of fixed points reproduces all the known physics of the $c = 1$ theory and respects T-duality.

4 The AdS minisuperspace from large N RG

We would like to know that in the context of our large N RG, which is formally dealing with the full superspace, is there really any limit in terms of the scale l that would confine us to the minisuperspace and reproduce the familiar minisuperspace results computed by the continuum Liouville theory or by the free fermionic matrix model? We would also like to know that if such a limit exists, then how the full quantum wave function is differing from the minisuperspace wave function away from this limit. This will help us in understanding the limitations of such a truncation of superspace in computing the actual quantum wave function of the universe. We will see that the minisuperspace physics in $c = 1$ matrix model corresponds to quantum mechanics of N degrees of freedom, the eigenvalues. Here, we will show that the large N RG acting on matrix quantum mechanics observables on the space of eigenvalues correctly captures the familiar minisuperspace WdW equation. This tests the correctness of the approach in deriving the WdW constraint.

To be more specific, let us consider the one point function of loop operator in the space of $N + 1$ eigenvalues from the diagonal matrices $\Lambda_{N+1}(\hat{x}) = \Omega^{-1}\phi_{N+1}(\hat{x})\Omega$, $\Omega \in SU(N)$ with periodic boundary condition $\lambda(0) = \lambda(2\pi R)$, given by

$$\langle \mathrm{Tr} e^{l\lambda_{N+1}(\hat{x})}\rangle = \langle e^{l\lambda(\hat{x})}\rangle^{N+1} . \tag{4.1}$$

We then integrate out one eigenvalue $\lambda_0(\hat{x})$ adjusting the matrix couplings according to double scaling limit, thus keeping g_s fixed. This generates a difference equation

$$\langle \mathrm{Tr} e^{l\Lambda_{N+1}(\hat{x})}\rangle_{\eta_i} - \langle \mathrm{Tr} e^{l\Lambda_N(\hat{x})}\rangle_{\eta'_i} = (\langle e^{l\lambda_0(\hat{x})}\rangle_{\eta'_i} - 1)\langle \mathrm{Tr} e^{l\Lambda_N(\hat{x})}\rangle\,, \tag{4.2}$$

where, η'_i are the renormalized matrix couplings given by

$$\begin{aligned} g' &= g(1 - 1/2N)\,, \quad \beta_g = -g/2\,, \\ M'^2 &= M^2\,, \quad \beta_{M^2} = 0\,. \end{aligned} \tag{4.3}$$

Note that, the beta functions above are too trivial to capture the nontrivial $c = 1$ fixed point and thus only have the trivial gaussian fixed point. This happens because in the space of the eigenvalues one practically works with the gaussian matrix model.

Now the left hand side of the difference equation can be interpreted as the Callan-Symanzik operator

$$\langle \mathrm{Tr} e^{l\Lambda_{N+1}(\hat{x})} \rangle_{\eta_i} - \langle \mathrm{Tr} e^{l\Lambda_N(\hat{x})} \rangle_{\eta_i'} = \Big(\frac{\partial}{\partial N} - \delta g \frac{\partial}{\partial g} \Big) \langle \mathrm{Tr} e^{l\Lambda_N(\hat{x})} \rangle_{\eta_i'} \,, \tag{4.4}$$

that simplifies to

$$\begin{aligned} &\Big(\frac{\partial}{\partial N} - \delta g \frac{\partial}{\partial g} \Big) \langle e^{l\lambda(\hat{x})} \rangle_{\eta_i'}^N \\ &= N \Big\langle - \int dt \big[\lambda(t) \frac{1}{2} (-\partial_t^2 + M^2) \lambda(t) - g \lambda^3(t) \big] e^{l\lambda(\hat{x})} \Big\rangle_{\eta_i'} \langle e^{l\lambda(\hat{x})} \rangle_{\eta_i'}^{N-1} \,. \end{aligned} \tag{4.5}$$

Now, just like in free fermionic theory, rescaling $\lambda_N \to \frac{\lambda_N}{\sqrt{N}}$ cubic interaction term becomes $O(1/N^2)$ and we focus on the top of the quadratic potential. Simultaneously we rescale $l \to \sqrt{N} l$ keeping the arguments of the exponentials unchanged. This makes

$$\Big(\frac{\partial}{\partial N} - \delta g \frac{\partial}{\partial g} \Big) \langle \mathrm{Tr} e^{l\Lambda_N(\hat{x})} \rangle_{\eta_i'} = -\frac{1}{2} M^2 \partial_l^2 \langle \mathrm{Tr} e^{l\Lambda_N(\hat{x})} \rangle_{\eta_i'} \,. \tag{4.6}$$

Plugging this in (4.2) we have the minisuperspace WdW equation

$$[-\frac{1}{2} M^2 l^2 \partial_l^2 + 4\mu l^2] \psi(l) = 0 \,. \tag{4.7}$$

Here the wave function $\psi(l)$ is given by the one point function $\langle \mathrm{Tr} e^{l\lambda_N(\hat{x})} \rangle_{\eta_i'}$. Comparing (4.7) with the standard minisuperspace WDW equation (2.4), the world sheet cosmological constant is given by

$$4\tilde{\mu} = 8\mu/M^2 = \frac{2}{M^2} (1 - \langle e^{l\lambda_0} \rangle_{\eta_i'}) \,. \tag{4.8}$$

Now to determine the sign of the cosmological constant, recall [43, 44, 45] that in the free fermionic theory the single particle wave functions in the inverted harmonic oscillator potential are supported at $-\infty \leq \lambda \leq 0$ (note that, here we have opposite sign of the cubic interaction term as that of [45]). Also the cosmological constant being of $O(1/M^2)$ is small. Thus $\tilde{\mu} \gtrsim 0$ ($\tilde{\mu} \to 0$ as $\lambda_0 \to 0$ and $g \to 0$) and comparing (4.7) with (2.4), we see that the large N RG of the matrix model observables defined on the space of eigenvalues indeed gives rise to AdS quantum mechanics, as would be expected from free fermionic theory. This tests the validity of our approach for the WdW cosmology we are addressing in this paper.

5 The modified Wheeler-de Witt constraint

So far we have discussed the setup for large N RG analysis giving rise to flow equations that have nontrivial $c = 1$ fixed points. We also have discussed the minisuperspace limit of the large

N RG flow that gives rise to the familiar AdS QM computed by free fermion representation of the $c = 1$ matrix model. Let us now move towards the computation of the wave function by deriving the modified Wheeler-de Witt (WdW) equation that dynamically determines the required wave function. In fact, the Callan-Symanzik equation for the one point function of the loop operator with Dirichlet boundary condition itself gives rise to this constraint.

5.1 Integrating over the vectors

Let us consider the expectation value of the loop operator at a fixed $\hat{x}$ with respect to the hermitian matrix quantum mechanics path integral with a cubic potential and couplings $\eta_i = g, M^2$

$$\psi(l) = \frac{1}{N} \left\langle \text{Tr } e^{l\phi(\hat{x})} \right\rangle_{\eta_i} . \tag{5.1}$$

As before, the matrices obey periodic boundary condition. Considering the parametrization (3.7) and expanding the operator, we have

$$\langle \text{Tr } e^{l\phi_{N+1}(\hat{x})} \rangle_{\eta_i} = \int D\phi_N(t)\ e^{-\tilde{S}_N(\phi)} I_{v(t)} \sum_{k=0}^{\infty} \frac{l^k}{k!} [\text{Tr}\phi_N^k(\hat{x}) + k v_N^*(\hat{x}) \phi_N^{k-2}(\hat{x}) v_N(\hat{x})] . \tag{5.2}$$

Here $\tilde{S}_N(\phi)$ is the v, v^* independent part of the reduced action given by

$$\tilde{S}_N(\phi) = N\ \text{Tr}(1 + \frac{1}{N}) \Big[\frac{\dot{\phi}_N^2}{2} + \frac{M^2}{2} \phi_N^2 - \frac{g}{3} \phi_N^3 \Big] . \tag{5.3}$$

The integral over the vectors $I_{v(t)}$ is the usual one as in (3.9)

$$I_{v(t)} = \int dv_N^*(t) dv_N(t) \exp[- \int_0^{2\pi R} dt\ v_N^*(t)(-\partial_t^2 + M^2 - g\phi_N(t)) v_N(t)] . \tag{5.4}$$

As we have discussed already, this is basically similar to the integral over the bosonic quark loops in Veneziano type QCD with a large color and flavor ($N = N_f$). The change in the expectation value of the loop operator due to the renormalization of the couplings η_i can now be expressed as

$$\langle \text{Tr} e^{l\phi_{N+1}(\hat{x})} \rangle_{\eta_i} - \langle \text{Tr} e^{l\phi_N(\hat{x})} \rangle_{\eta_i'} = l \Big\langle \frac{1}{\Sigma}\ I_{v(t)} v_N^*(\hat{x}) e^{l\phi_N(\hat{x})} \phi_N^{-1}(\hat{x}) v_N(\hat{x}) \Big\rangle_{\eta_i'} . \tag{5.5}$$

Here Σ represents the diagrammatic expansion of $I_{v(t)}$ that eventually renormalizes the bare action (recall (3.16)). Thus the expectation value of any operator $\mathcal{O}$ with respect to the renormalized couplings η_i' can formally be given by

$$\langle \mathcal{O} \rangle_{\eta_i'} = \int D\phi_N\ \mathcal{O}\ e^{-\tilde{S}_N[\phi]}\ \Sigma . \tag{5.6}$$

We will now closely examine the meaning of the both sides of the equation (5.5).

5.2 The Callan-Symanzik operator and the modified WdW constraint

The change in the expectation value in (5.5) is essentially the Callan Symanzik operator in discrete form, acting on the one point function of the loop operator. Consider the discrete relation

$$\langle \mathrm{Tr} e^{l\phi_{N+1}(\hat{x})}\rangle_{\eta_i} - \langle \mathrm{Tr} e^{l\phi_N(\hat{x})}\rangle_{\eta_i'} = \left[\frac{\partial}{\partial N} - \delta\eta_i \frac{\partial}{\partial \eta_i}\right] \langle \mathrm{Tr} e^{l\phi_N(\hat{x})}\rangle_{\eta_i} \, . \tag{5.7}$$

Now recalling (3.3) defining the beta functions $\beta_{\eta_i} = N\delta\eta_i$ in the large N RG, and using the identity $\frac{\partial}{\partial N}\langle \mathrm{Tr}\ e^{l\phi_N(\hat{x})}\rangle_{\eta_i} = (N\partial/\partial N + 1)\frac{1}{N}\langle \mathrm{Tr}\ e^{l\phi_N(\hat{x})}\rangle_{\eta_i}$, the left hand side of (5.5) can clearly be written as a Callan-Symanzik operator acting on $\psi(l, \hat{x})$

$$\langle \mathrm{Tr} e^{l\phi_{N+1}(\hat{x})}\rangle_{\eta_i} - \langle \mathrm{Tr} e^{l\phi_N(\hat{x})}\rangle_{\eta_i'} = \left[N\frac{\partial}{\partial N} - \beta_{\eta_i} \frac{\partial}{\partial \eta_i} + 1\right] \frac{1}{N}\langle \mathrm{Tr} e^{l\phi_N(\hat{x})}\rangle_{\eta_i} \, . \tag{5.8}$$

However, the wave function is not completely annihilated by the Callan-Symanzik operator. The nonvanishing right hand side is essentially computing the anomaly in a nontrivial way. The origin of this anomaly lies in the fact that the loops of same length can continuously be mapped into each other by $2D$ diffeomorphism which acts like a gauge symmetry. The WdW operator acts as a gauge fixing condition.

We will compute the anomaly below and will see that it is proportional to $\phi_N^{-1}(\hat{x})$ which is singular in the small field approximation. The singularity can be tackled by taking derivatives with respect to l on both sides of the Callan-Symanzik equation. The right hand side then simplifies to a differential operator acting on the one point function. In this form, the operator, which annihilates the one point function of the loop operator, gives nothing but the *modified WdW constraint*. Physically this constraint governs the dynamics of loops of arbitrary shape with arbitrary length fluctuations as compared to the usual (minisuperspace) WdW constraint that only deals with the dynamics of a spherical loop. We will discuss in detail in what sense this constraint is 'modified', *i.e.* 'beyond minisuperspace', compared to the usual minisuperspace WdW constraint and it's implications. In this paper, we will show that the very existence of such a general 'beyond minisuperspace' constraint in low dimensional noncritical string theory is the key to probe question like inflation and topology change.

5.3 Computing the anomaly

Let us now go back to the computation of the anomaly term. As before, let us rescale the vectors as $v_i \to v_i/\sqrt{2\pi R}$ and Fourier decompose all the vector and matrix degrees of freedom. We now perform the following small field expansion of the non-gaussian part of (5.4) containing the interaction vertex $\mathcal{O}^{v^*v}_{m-l}(g,\phi) = g\phi_{m-l}$

$$l\Big\langle\frac{1}{\Sigma}\ I_{v(t)}v^*(\hat{x})e^{l\phi_N(\hat{x})}\phi_N(\hat{x})v(\hat{x})\Big\rangle_{\eta_i'} = l\Big\langle\frac{1}{\Sigma}\ \int\frac{\prod_m dv_m^* dv_m}{(2\pi R)^{N+1}}\ \sum_{k,l} e^{\frac{i(-k+l)}{R}\hat{x}}v_k^*(e^{l\phi_N(\hat{x})}\phi_N^{-1}(\hat{x}))v_l$$
$$\Big(1+g\sum_{m,n}v_m^*\phi_{m-n}v_n+\frac{g^2}{2!}\sum_{m,n}\sum_{p,q}v_m^*\phi_{m-n}v_n\ v_p^*\phi_{p-q}v_q+\dots\Big)\ \exp[-\sum_m v_m^*(m^2/R^2+M^2)v_m]\Big\rangle_{\eta_i'}. \tag{5.9}$$

We then contract all possible v, v^* pairs in the expansion that (upon acting on the remaining gaussian part of the integration) extracts inverse propagator $(\mathcal{O}_{mn}^{v^*v})^{-1} = (\frac{mn}{R^2}+M^2)^{-1}\delta_{m,n}$. The gaussian integration I_0 gives an overall pre-factor that neutralizes a part of the denominator. The contractions nontrivially mixes up the momentum modes at interaction vertices generating complicated interaction terms in the diagrammatic expansion. However, the Fourier modes of the singular term $e^{l\phi_N(\hat{x})}\phi_N^{-1}(\hat{x})$ do not get mixed with that of other terms. Hence the term is effectively decoupled from the rest of the expansion and does not receive any correction. The rest of the term with quantum corrections can finally be cast as a series summation of differential operators which we will explain below. This decoupling property of the singular term eventually gives back the one point function of the loop operator and is thus crucial in closing the WdW constraint. Here for simplicity we do not show the Fourier decomposition of the term. The resulting diagrammatic expansion due to the integration over vectors with insertion of the loop operator at the Dirichlet boundary is

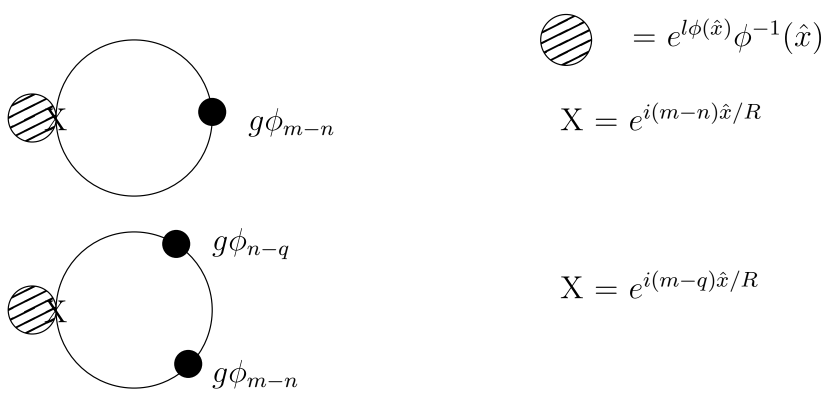

Figure 5: Diagrams representing $O(g)$ and $O(g^2)$ terms in (5.10).

$$
\begin{aligned}
&\langle \mathrm{Tr} e^{l\phi_{N+1}(\hat{x})}\rangle_{\eta_i} - \langle \mathrm{Tr} e^{l\phi_N(\hat{x})}\rangle_{\eta_i'} = \frac{l}{2\pi R}\Big\langle \frac{1}{\tilde{\Sigma}}\Big[\mathrm{Tr}\Big(\frac{\pi R}{M}\ \coth\pi MR\ e^{l\phi_N(\tilde{x})}\phi_N^{-1}(\hat{x})\Big)\\
&+g\ \mathrm{Tr}\Big(e^{l\phi_N(\hat{x})}\phi_N^{-1}(\hat{x})\sum_{m,n}\frac{e^{i(-n+m)\hat{x}/R}\phi_{m-n}}{(n^2/R^2+M^2)(m^2/R^2+M^2)}\Big)\\
&+\frac{g^2}{2}\ \mathrm{Tr}\Big(e^{l\phi_N(\hat{x})}\phi_N^{-1}(\hat{x})\sum_{m,n,q}\frac{e^{i(-q+m)\hat{x}/R}\phi_{m-n}\phi_{n-q}}{(m^2/R^2+M^2)(n^2/R^2+M^2)(q^2/R^2+M^2)}\Big)\\
&+\frac{g^3}{6}\ \mathrm{Tr}\Big(e^{l\phi_N(\hat{x})}\phi_N^{-1}(\hat{x})\sum_{m,n,q,k}\frac{e^{i(-k+m)\hat{x}/R}\phi_{m-n}\phi_{n-q}\phi_{q-k}}{(m^2/R^2+M^2)(n^2/R^2+M^2)(q^2/R^2+M^2)(k^2/R^2+M^2)}\Big)\\
&+\ldots\Big]\Big\rangle_{\eta_i'}\,,
\end{aligned}
\tag{5.10}
$$

where $\tilde{\Sigma}$ denotes the non-Gaussian part of the diagrammatic expansion of the vector integral

$$
\Sigma = \frac{I_0}{(2\pi R)^N}\ \tilde{\Sigma}\,, \tag{5.11}
$$

with I_0 representing the Gaussian part. It is now convenient to inverse Fourier transform the expansion term by term and then evaluate the Feynman diagrams by summing up the series with exponentials. Thus in terms of the target space diagrams, the small field expansion for the anomaly becomes

$$
\begin{aligned}
&\langle \mathrm{Tr} e^{l\phi_{N+1}(\hat{x})}\rangle_{\eta_i} - \langle \mathrm{Tr} e^{l\phi_N(\hat{x})}\rangle_{\eta_i'}\\
&= \frac{l}{2\pi R}\Big\langle\frac{1}{\tilde{\Sigma}}\mathrm{Tr}\Big[e^{l\phi_N(\hat{x})}\phi_N^{-1}(\hat{x})\Big\{\frac{\pi R}{M}\coth\pi MR\\
&+\frac{g}{M^4}\int\frac{dt}{2\pi R}\sum_m\Big(1+\frac{m^2}{R^2M^2}\Big)^{-1}e^{im(\hat{x}-t)/R}\phi(t)\sum_n\Big(1+\frac{n^2}{R^2M^2}\Big)^{-1}e^{in(t-\hat{x})/R}\\
&+\frac{g^2}{2M^4}\int\frac{dt_1dt_2}{(2\pi R)^2}\sum_m\Big(1+\frac{m^2}{M^2R^2}\Big)^{-1}e^{im(\hat{x}-t_1)/R}\phi(t_1)\sum_n\frac{e^{in(t_1-t_2)/R}}{n^2/R^2+M^2}\phi(t_2)\\
&\sum_q\Big(1+\frac{q^2}{M^2R^2}\Big)^{-1}e^{iq(t_2-\hat{x})/R}\\
&+\frac{g^3}{6M^4}\int\frac{dt_1dt_2dt_3}{(2\pi R)^3}\sum_m\Big(1+\frac{m^2}{M^2R^2}\Big)^{-1}e^{im(\hat{x}-t_1)/R}\phi(t_1)\sum_n\frac{e^{in(t_1-t_2)/R}}{n^2/R^2+M^2}\phi(t_2)\\
&\frac{e^{iq(t_2-t_3)/R}}{q^2/R^2+M^2}\phi(t_3)\sum_k\Big(1+\frac{k^2}{M^2R^2}\Big)^{-1}e^{ik(t_3-\hat{x})/R}+\ldots\Big\}\Big]\Big\rangle_{\eta_i'}\ .
\end{aligned}
\tag{5.12}
$$

Now to evaluate the Feynman diagrams, we use the open string moduli $\hat{x}$ to replace the inverse propagator by a differential operator of $\hat{x}$

$$\mathcal{O}_{\hat{x}} \equiv \Big(1 - \frac{1}{M^2}\frac{\partial^2}{\partial \hat{x}^2}\Big)^{-1} . \tag{5.13}$$

This simplifies the summation over exponentials in the above expression to delta functions $\delta(t_i - \hat{x})$ that forces any matrix quantum mechanics time t_i (matter) to be at $\hat{x}$, which is basically reflecting the fact that we are talking about Dirichlet boundaries that are introducing explicit dependence on the open string moduli due to nonperturbative effects coming from all the N^2 quantum mechanical degrees of freedom. Note that replacing the inverse propagator by (5.13) is not only consistent formally but also can be seen to be in agreement as an expansion in the large M limit for Dirichlet boundaries. In the extreme $M \to \infty$ limit, the explicit open string moduli dependence is washed out. After performing the delta function integrals over the matrix quantum mechanics time, the diagrammatic expansion can now be written as

$$\begin{aligned}
&\langle \mathrm{Tr} e^{l\phi_{N+1}(\hat{x})}\rangle_{\eta_i} - \langle \mathrm{Tr} e^{l\phi_N(\hat{x})}\rangle_{\eta'_i} \\
&= \frac{l}{2\pi R}\Big\langle \frac{1}{\tilde{\Sigma}}\Big[\frac{\pi R}{M}\coth \pi MR\ \mathrm{Tr}(e^{l\phi_N(\hat{x})}\phi_N^{-1}(\hat{x})) \\
&+ \frac{2\pi R g}{M^4}\mathrm{Tr}(e^{l\phi_N(\hat{x})}\phi_N^{-1}(\hat{x})\ \mathcal{O}_{\hat{x}}\phi(\hat{x})\mathcal{O}_{\hat{x}}) + \frac{\pi R}{M}\coth \pi MR\ \frac{g^2}{2M^4}\ \mathrm{Tr}(e^{l\phi_N(\hat{x})}\phi_N^{-1}(\hat{x})\ \mathcal{O}_{\hat{x}}\phi^2(\hat{x})\mathcal{O}_{\hat{x}}) \\
&+ \frac{2\pi R g^3}{6M^8}\ \mathrm{Tr}(e^{l\phi_N(\hat{x})}\phi_N^{-1}(\hat{x})\mathcal{O}_{\hat{x}}\phi(\hat{x})\mathcal{O}_{\hat{x}}\phi(\hat{x})\mathcal{O}_{\hat{x}}\phi(\hat{x})\mathcal{O}_{\hat{x}}) + ...\Big]\Big\rangle_{\eta'_i} ,
\end{aligned} \tag{5.14}$$

The above diagrammatic expansion can now be arranged according to even and odd powers of matrix couplings which sum up to even and odd series of $\frac{g}{M^2}\mathcal{O}_{\hat{x}}\phi(\hat{x})$, thus computing the anomaly of the form

$$\langle \mathrm{Tr} e^{l\phi_{N+1}(\hat{x})}\rangle_{\eta_i} - \langle \mathrm{Tr} e^{l\phi_N(\hat{x})}\rangle_{\eta'_i} = \frac{l}{2\pi R}\langle \tilde{\Sigma}^{-1}\ \mathrm{Tr}[(e^{l\phi_N(\hat{x})}\phi_N^{-1}(\hat{x}))(S_1 + S_2)]\rangle , \tag{5.15}$$

where the diagrams with even powers in coupling are given by

$$\begin{aligned}
S_1 &= \frac{\pi R}{M}\coth \pi MR\big[1 + \frac{1}{2!}\frac{1}{M^2}\ \mathcal{O}_{\hat{x}} g^2\phi^2(\hat{x})\frac{1}{M^2}\mathcal{O}_{\hat{x}} \\
&\quad + \frac{1}{4!}\frac{1}{M^2}\mathcal{O}_{\hat{x}} g\phi(\hat{x})\frac{1}{M^2}\mathcal{O}_{\hat{x}} g^2\phi^2(\hat{x})\frac{1}{M^2}\mathcal{O}_{\hat{x}} g\phi(\hat{x})\frac{1}{M^2}\mathcal{O}_{\hat{x}} + \dots\big] \\
&= \frac{\pi R}{M}\coth \pi MR\ \cosh(\frac{g}{M^2}\mathcal{O}_{\hat{x}}\phi(\hat{x})) ,
\end{aligned} \tag{5.16}$$

and the odd powers in coupling are given by

$$
\begin{aligned}
S_2 &= 2\pi R\Big[\frac{1}{M^2}\mathcal{O}_{\hat{x}} g\phi(\hat{x})\frac{1}{M^2}\mathcal{O}_{\hat{x}} + \\
&\quad \frac{1}{3!}\frac{1}{M^2}\mathcal{O}_{\hat{x}} g\phi(\hat{x})\ \frac{1}{M^2}\mathcal{O}_{\hat{x}} g\phi(\hat{x})\ \frac{1}{M^2}\mathcal{O}_{\hat{x}} g\phi(\hat{x})\ \frac{1}{M^2}\mathcal{O}_{\hat{x}} + \dots\Big] \\
&= 2\pi R\frac{1}{M^2}\mathcal{O}_{\hat{x}}\ \sinh\Big(\frac{g}{M^2}\mathcal{O}_{\hat{x}}\phi(\hat{x})\Big)\,. \qquad (5.17)
\end{aligned}
$$

Note that, adding up the series neatly to hyperbolic functions is consistent with the small field approximation as well as the large M limit for the Dirichlet boundaries.

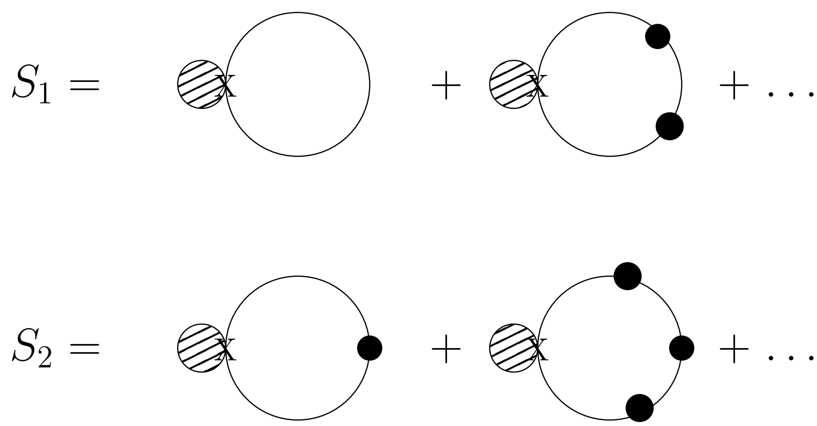

Figure 6: Diagrams contributing in S_1 and S_2.

5.4 The modified WdW constraint

Let us now differentiate both sides of (5.15) with respect to the length l of the loop to remove the singular ϕ_N^{-1} term from the right hand side. Also we do small field expansion of $\tilde{\Sigma}^{-1}$. Let us now recast powers of $\phi_N(\hat{x})$ to be multi derivatives of l on the loop operator. This closes the WdW constraint by factorizing the right hand side of (5.15) into a differential operator of l and $\hat{x}$ operating on the one point function of the loop operator and a modified WdW equation emerges. Thus

$$
\begin{aligned}
\frac{\partial}{\partial l}\Big[\Big\langle\frac{1}{l}\mathrm{Tr}\ e^{l\phi_{N+1}(\hat{x})}\Big\rangle_{\eta_i} - \Big\langle\frac{1}{l}\mathrm{Tr}\ e^{l\phi_N(\hat{x})}\Big\rangle_{\eta_i'}\Big] = &\quad \Big[\frac{1}{2M}\ \coth \pi M R \cosh\Big(\frac{g}{M^2}\ \mathcal{O}_{\hat{x}}\frac{\partial}{\partial l}\Big) \\
&+\frac{1}{M^2}\ \mathcal{O}_{\hat{x}}\ \sinh\Big(\frac{g}{M^2}\frac{\partial}{\partial l}\ \mathcal{O}_{\hat{x}}\Big)\Big]\ \langle \mathrm{Tr} e^{l\phi(\hat{x})}\rangle_{\eta_i'} + \dots\,, \qquad (5.18)
\end{aligned}
$$

where the ellipsis corresponds to the multi-trace or worm hole deformations that leads to topology changing amplitudes. These are redundant operators and are in relative $O(1/N)$ compared to the leading contribution and contribute to topology changing transitions. Using (5.8) the linear (dealing with loop of a single topology) generalized WdW equation can be written as

$$\Big[\Big(-\frac{1}{l^2}+\frac{1}{l}\frac{\partial}{\partial l}\Big)\Big(N\frac{\partial}{\partial N}-\beta_{\eta_i}\frac{\partial}{\partial\eta_i}+1\Big)-\frac{N}{2M}\coth\pi MR\cosh\Big(\frac{g}{M^2}\mathcal{O}_{\hat{x}}\frac{\partial}{\partial l}\Big)$$
$$-\frac{N}{M^2}\mathcal{O}_{\hat{x}}\sinh\Big(\frac{g}{M^2}\frac{\partial}{\partial l}\mathcal{O}_{\hat{x}}\Big)\Big]\frac{1}{N}\langle\mathrm{Tr}\ e^{l\phi_N(\hat{x})}\rangle_{\eta_i'}=0\,. \tag{5.19}$$

Clearly the WdW constraint is *modified* or *generalized*, as compared to the usual one, in the sense of being beyond minisuperspace (see the discussion in section 2). It captures effects of derivatives of all orders, which is more than the effect caught by a perturbative superspace (considering fluctuations on the Fermi surface in the free fermion language). Also the higher derivative effect captured here is for all genus wave function as compared to the genus zero result obtained in the perturbative superspace. Moreover, the scale factor l here contains all the modes of the metric fluctuation as compared to the presence of zero mode l_0 only in the dynamical Fermi sea picture. Note the difference in the l dependence of the coefficients of $V(\partial/\partial l_0)$ and $V'(\partial/\partial l_0)$ in (2.5) and that of the cosh and sinh terms in (5.19). This point needs to be understood. Another aspect of (5.19) being beyond minisuperspace is that, it has an in built way of determining the factor ordering ambiguity. However, as we mentioned above, the linear modified WdW constraint that governs the dynamics of loops of a particular topology, comes from the leading order contribution of $\tilde{\Sigma}^{-1}$. The sub-leading orders, beginning from a relative $O(1/N)$ compared to the single trace terms, give rise to the nonlinear part of the WdW constraint corresponds to dynamics of topological transitions between loops of different topologies. This is strictly a beyond minisuperspace process that can no way be captured within minisuperspace. We will return to this in the last section.

From now on let us write (5.19) in terms of the wave function (5.1). In order to reduce the dimension of the space of independent variables, let us consider the following ansatz for the Callan-Symanzik equation of the macroscopic loop variables

$$\big(N\frac{\partial}{\partial N}-\beta_{\eta_i}\frac{\partial}{\partial\eta_i}+1\big)\psi(l;N,\eta_i,R)=\big(l^2\mathcal{F}_1(\eta_i,N,R)+\mathcal{F}_2(\eta_i,N,R)\big)\ \psi(l;N,\eta_i,R)\,. \tag{5.20}$$

Here, $[\mathcal{F}_1]=[l^{-2}]$ and $\mathcal{F}_2$ is dimensionless. The ansatz is chosen keeping in mind that the wave function behaves somewhat like the exponential of the Euclidean action, the area l^2. From our discussion in the previous subsections we already know that the macroscopic loop has an anomaly in the Callan-Symanzik equation which demands that they are annihilated by the WdW operator that acts as a gauge fixing condition for them. Here, by considering this ansatz, we are actually assuming that we are looking at those macroscopic loops which not only have WdW constraint as a gauge fixing condition, also they themselves are invariant under scale changes.

So that, for them the anomaly basically simplifies into some kind of *anomalous dimension* (a number). This reduces the space of independent variables to the dynamical variables only, the length of the loop l. As we mentioned before, due to nonperturbative effect coming from all the N^2 quantum mechanical degrees of freedom, the wave function has explicit dependence on the open string modulus $\hat{x}$ which acts as another dynamical variable. Now our job is to plug in (5.20) into (5.19) and solve for the wave function that as well determines $\mathcal{F}_1$ and $\mathcal{F}_2$ in a self consistent way. The functions $\mathcal{F}_1$ and $\mathcal{F}_2$ keep the information about the $c = 1$ fixed point of the matrix quantum mechanics around which the solution is being studied.

Let us consider another simplification. Since the time direction is compact, $\hat{x} = 0$ can be identified with $\hat{x} = 2\pi R$. Therefore one can assume the $\hat{x}$ dependence of the wave function $\psi(l, \hat{x})$ to be periodic. The simplest choice, that is nonzero at $\hat{x} = 0$, is plane waves $\psi(\hat{x}) = \cos(\frac{n}{R}\hat{x})$ with quantized momenta. Thus,

$$\mathcal{O}_{\hat{x}}\psi(\hat{x}) = \lambda_n\ \psi(\hat{x})\,, \qquad \lambda_n = \frac{M^2}{M^2 + n^2/R^2}\,. \tag{5.21}$$

Thus considering (5.20), (5.19) and (5.21), the beyond minisuperspace WdW equation for the one point function of loop operator becomes

$$\Big[2l\ A\cosh\big(Q\ \lambda_n\frac{d}{dl}\big) + 2l\ B\lambda_n \sinh\big(Q\ \lambda_n\frac{d}{dl}\big) - (l^2\mathcal{F}_1 + \mathcal{F}_2)\frac{d}{dl} - 2l\mathcal{F}_1\Big]\psi(l) = 0\,, \tag{5.22}$$

where,

$$A = \frac{N}{2M}\coth\pi MR\,, \quad B = \frac{N}{M^2}\,, \quad Q = \frac{g}{M^2}\,. \tag{5.23}$$

It is important to note here that all the parameters A,B,Q have $M \to -M$ symmetry (see the discussion under $c = 1$ fixed point in the previous section).

To study the solution of (5.22) at different scales, especially at small and large scale limits, it will be useful to change the dependent variable to one point function with one puncture

$$\Psi(l, \hat{x}) = \frac{\psi(l, \hat{x})}{l} = \frac{1}{N}\langle \mathrm{Tr}\frac{e^{l\phi_N(\hat{x})}}{l}\rangle_{\eta_i'}\,. \tag{5.24}$$

This basically defines a closed Dirichlet boundary of length l with one marked point on it. Using the following identity

$$e^{\pm Q\frac{d}{dl}}\ \frac{\psi(l)}{l} = \frac{1}{l}\ e^{\pm Q\frac{d}{dl}}\ \psi(l) + \frac{\psi(l)}{l \pm Q} \tag{5.25}$$

in (5.22), the WdW equation for the one point function with one puncture can be rewritten as

$$\Big[2l\ A\cosh\big(Q\ \lambda_n\frac{d}{dl}\big) + 2l\ B\lambda_n \sinh\big(Q\ \lambda_n\frac{d}{dl}\big) - (l^2\mathcal{F}_1 + \mathcal{F}_2)\frac{d}{dl} - 2l\mathcal{F}_1 - \frac{2l^2(Al - BQ\lambda_n^2)}{l^2 - Q^2\lambda_n^2}\Big]\Psi(l) = 0\,. \tag{5.26}$$

This actually induces a polynomial potential term in the WdW equation that brings in the notion of a special scale $\tilde{l}(n) = Q\lambda_n$ separating the UV and IR physics. From $2D$ cosmology point of view this is naturally a inverse Hubble scale H^{-1} for the problem. In the next section we will discuss the significance of this scale in detail and study physics of the solution to (5.26) with respect to it.

6 The cosmological implications of the wave function

In the previous section we have seen that the modified WdW equation (5.26) for one point function with one puncture has a polynomial potential that naturally brings in the notion of a special scale $\tilde{l}(n) = Q\lambda_n$ for the problem. In this section we will study the cosmological implication of this scale and the study the small and large scale behavior of the solution to (5.26) with respect to this scale.

6.1 The inverse Hubble scale

Let us consider a rough qualitative analysis of the shape of the polynomial potential in (5.26) with respect to the scale l. Here one can expand the hyperbolic functions of differential operator and write down the potential with respect to the quadratic differential term $(-l^2 \frac{\partial^2}{\partial l^2})$ in the Liouville metric $d\varphi = dl/l$. Such an expansion is possible considering $Q = g/M^2$ to be small, which is definitely the case around a $c = 1$ fixed point that has small g and for Dirichlet boundary that has large M. The potential is given by

$$V(l,n) = \frac{2l^2}{\tilde{l}(n)^2}\Big(\frac{\mathcal{F}_1}{A} + \frac{l(l - \frac{B\lambda_n}{A}\tilde{l}(n))}{(l^2 - \tilde{l}(n)^2)} + 1\Big)\,. \tag{6.1}$$

Here n is related to the momentum modes of matter. Since $[\mathcal{F}_1] = [A] = [l^{-2}]$, the ratio $\mathcal{F}_1/A$ is purely a number. Now the structure of the potential depends on the size of the quantity $B\lambda_n/A$. For Dirichlet boundaries

$$B\lambda_n/A = 2M\tanh(\pi MR)/(M^2 + \frac{n^2}{R^2}) \sim 2/M << 1\,. \tag{6.2}$$

This shows that for $l << \tilde{l}(n)$, the potential $V(l,n)$ looks like a barrier in a tunneling problem.

This is more clear from the form of the potential (6.1) along with the constraint (6.2) in $\hat{x}$ space

$$V(l,\hat{x}) = V_{local} + \frac{g^2}{l^2}\frac{\pi R}{\sqrt{M^2 - g^2/l^2}}\frac{\cosh\sqrt{M^2 - g^2/l^2}(\pi R - \hat{x})}{\sinh\sqrt{M^2 - g^2/l^2}\;\pi R}(2 - 4\pi\tanh\pi MRl/g)\,, \tag{6.3}$$

where

$$V_{local} = \frac{2l^2}{g^2}[2-(C^{-1}+2Ce^{-C})][M^2\delta(\hat{x})+\delta''(\hat{x})]+(2-4M\tanh\pi MRl/g)\delta(\hat{x})\,, \qquad (6.4)$$

being proportional to $\delta(\hat{x})$ can be neglected by considering $\hat{x}\neq 0$. Thus the magnitude of the matrix quantum mechanics parameters around the $c=1$ fixed point as compared to the smallness or largeness of l and the form of $\tilde{V}=V-V_{local}$ controls the behavior. Let us consider that, as $l\to 0$, $g/l\ll M$. Then, assuming $\hat{x}\sim\pi R\sim 1$, $\tilde{V}$ simplifies to

$$\tilde{V}(l\to 0)\sim\Big(Me^{-M}(\frac{2g^2}{l^2M^2})-4\pi e^{-M}\frac{g}{lM}\Big)\to 0\,. \qquad (6.5)$$

On the other hand $\tilde{V}(l\to\infty)\to-\infty$ and $\tilde{V}\to 0$ at a scale $l\to O(g/M)>\tilde{l}\sim O(g/M^2)$. This provides the tunneling barrier. Recall that the special scale $\tilde{l}$ occurs before the potential goes to zero.

This indicates that for small scale $l<\tilde{l}$, the $1D$ closed universe with homogenous matter that we are considering, presumably admits a tunneling type of wave function that creates the universe from nothing by Euclidean tunneling. In cosmological models of $4D$ gravity such tunneling wave function have been shown to predict inflation [5]. Whether the tunneling actually predicts an inflationary scenario depends on the scale to which universe tunnels to. If the scale expands exponentially in a short time, then the universe undergoes inflation. At large scale $l>\tilde{l}$ the potential approaches $-\infty$ and hence the wave function of the universe presumably admits outgoing plane wave that gives an expanding de Sitter universe in the far future. In the following subsections we will show that the phase $l<\tilde{l}$ will be the phase at which universe undergoes inflation and the phase $l>\tilde{l}$ will be the far future phase of an expanding universe which is already hugely inflated. The region, where $l=\tilde{l}$, actually gives the hot big bang.

Thus the scale $\tilde{l}(n)$ acts like an inverse Hubble scale H^{-1} that separates the small and large scale behavior of the wave function. For $c=1$ matrix model, it is at very short distance as would be appropriate for an inflationary scenario. Note that the nonperturbative effects due to the N^2 quantum mechanical degrees of freedom is introducing explicit open string moduli dependence of the Hubble scale. Though the moduli dependence does not directly affect the inflationary scenario, as one always has a tunneling wave function solution for any sensible constant H, its explicit dependence determines the geometry of the vacuum one tunnels to. The matter dependence of this scale can be written down by inverse Fourier transforming $\tilde{l}(n)$ as

$$\tilde{l}(\hat{x})=H(\hat{x})^{-1}=\frac{\pi gR}{M}\frac{\cosh(\pi MR-M\hat{x})}{\sinh(\pi MR)}\,, \qquad 0\leq\hat{x}\leq 2\pi R\,. \qquad (6.6)$$

This is the scale factor corresponding to a de Sitter geometry of the world sheet

$$ds^2 = dX^2 - \frac{g^2R^2}{M^2}\cosh^2(\pi MR - MX)\, d\varphi^2\,. \tag{6.7}$$

Recall that the world sheet matter $X(\sigma)$ is mapped to the matrix quantum mechanics time through the delta function in the Dirichlet boundary condition (2.3). The issue of the Lorentzian signature is related to the fact that the large N RG with N^2 matrix quantum mechanics degrees of freedom captures the de Sitter sign of the cosmological constant ($\mu < 0$) in the WdW equation (like the supercritical case) instead of the usual AdS sign ($\mu > 0$) of the $c = 1$ quantum mechanics. This effectively implies an imaginary slope of linear dilaton ($\gamma^2 < 0\,, \gamma \to i\gamma\,, Q = \gamma/2 \to iQ$) and hence indirectly implies growth of number of space-time dimensions. On the other hand the exponential potential wall $e^{\gamma\varphi}$ remains the same in the WdW equation. Hence with $\gamma \to i\gamma$, the Liouville field also becomes imaginary $\varphi \to -i\varphi$. This takes care of the reality of the Liouville action and gives rise to a Lorentzian metric.

6.2 The small scale behavior: inflation

Let us now solve for the small scale behavior of the wave function. For $l < \tilde{l} = Q\lambda_n$, The polynomial term can be expanded as

$$P(l) = \frac{2l^2}{\tilde{l}^2}(Al - B\lambda_n\tilde{l})\sum_{m=0}^{\infty}\frac{l^{2m}}{\tilde{l}^{2m}}\,. \tag{6.8}$$

As indicated by the potential (6.1), the wave function in the region $l < \tilde{l}$ can have tunneling through the barrier. Thus an appropriate trial solution satisfying (5.26) would be

$$\Psi(l) \sim a(N,\eta_i)e^{-\Delta(\eta_i)l^2} \tag{6.9}$$

Later we will argue that such a choice of trial function is a unique choice fixed by the initial profile of the scale fluctuation $\Psi(0 + \delta l)$. Now using the identities

$$\begin{aligned}\cosh(\tilde{l}\frac{d}{dl})e^{-\Delta l^2} &= \cosh(2\Delta\tilde{l}l)e^{-\Delta(l^2+\tilde{l}^2)}\,,\\ \sinh(\tilde{l}\frac{d}{dl})e^{-\Delta l^2} &= \sinh(2\Delta\tilde{l}l)e^{-\Delta(l^2+\tilde{l}^2)}\,,\end{aligned} \tag{6.10}$$

and expanding the hyperbolic functions in power series of l (for $\tilde{l} < 1$ *i.e.* $g < M^2$ and for a small l), we set the coefficients of all the powers of l to be zero. Thus we see that $\Psi(l) \sim a(N,\eta_i)e^{-\Delta(\eta_i)l^2}$ is a solution of (5.26) with the spectrum Δ determined by

$$
\begin{aligned}
&e^{-\Delta\tilde{l}^2} + \frac{\Delta\mathcal{F}_2}{A} - \frac{\mathcal{F}_1}{A} = 0\,,\\
&\Delta^2\ \tilde{l}^2\ e^{-\Delta\tilde{l}^2} + \frac{\Delta\mathcal{F}_1}{2A} + \frac{1}{2\tilde{l}^2} = 0\\
&\frac{(2\Delta\tilde{l})^n}{n!}e^{-\Delta\tilde{l}^2} + (-1)^n\frac{1}{\tilde{l}^n} = 0\,,\quad n \neq 0, 2\,.
\end{aligned}
\tag{6.11}
$$

Here we have three equations and three unknowns $\Delta, \mathcal{F}_1$ and $\mathcal{F}_2$. The equations (6.11) indicate that any solution for the spectrum Δ should go somewhat like Lambert$W$ function. For Lambert$W$ functions the function itself is in the same footing as it's exponential. Hence such a solution for the spectrum Δ will be a good candidate to actually perform the cancelation of the hyperbolic function by a polynomial (6.8) needed to satisfy equation (5.26). Thus solution to the spectrum Δ can be written as $\Delta = C/\tilde{l}^2$, where the number C is given by the three sets of relations in combined form

$$
e^{-C} = \frac{n!(-1)^{n+1}}{(2C)^n} = -\Big(\frac{1}{2C^2} + \frac{\mathcal{F}_1}{2AC}\Big) = \frac{\mathcal{F}_1}{A} - \frac{C\mathcal{F}_2}{A\tilde{l}^2}\,,\qquad n \neq 0, 2\,.
\tag{6.12}
$$

Now solving these relations we have

$$
\begin{aligned}
&C = -n\ \mathrm{LambertW}\Big(\frac{(-n!)^{1/n}}{2n}\Big)\,,\quad n \neq 0, 2\,,\\
&\mathcal{F}_1 = A\Big(-\frac{1}{C} - 2Ce^{-C}\Big)\,,\\
&\mathcal{F}_2 = A\ \tilde{l}^2\ \Big(-\frac{1}{C^2} - \frac{e^{-C}}{C} - 2e^{-C}\Big)\,.
\end{aligned}
\tag{6.13}
$$

Thus, Δ is given by

$$
\Delta = -\frac{n}{g^2}\ \mathrm{LambertW}\Big(\frac{(-n!)^{1/n}}{2n}\Big)\Big(M^2 + \frac{n^2}{R^2}\Big)^2\,,\quad n \neq 0, 2\,.
\tag{6.14}
$$

In order to give an idea, the table below illustrates different values of C corresponding to sample arguments, using (6.13).

n	C
1	-0.79+0.77i
3	1.5
4	0.62+1.13i
5	1.91
6	1.4 +1.27i
7	2.37
8	1.49+0.8i
9	2.84
10	2.68+1.26i
11	3.69

However, from (5.20) $\Delta, \mathcal{F}_1, \mathcal{F}_2$ and the amplitude of the wave function a satisfy additional constraints

$$\begin{aligned} \beta_{\eta_i}\frac{\partial \Delta}{\partial \eta_i} &= \mathcal{F}_1\,, \\ N\frac{\partial a}{\partial N} - \beta_{\eta_i}\frac{\partial a}{\partial \eta_i} + a &= \mathcal{F}_2 a\,. \end{aligned} \tag{6.15}$$

In writing down this constraints we considered the fact that all the four parameters $\Delta, \mathcal{F}_1, \mathcal{F}_2, a$ are independent of l and Δ does not have any explicit N dependence as indicated by the solution (6.14). Obviously the second one of the two relations in (6.15) determines the unknown parameter a. Now we need to check whether the other one is really a redundant condition or a serious constraint on the already determined parameters. To check this, let us recall that around $c = 1$ fixed point we have $h = 0$, *i.e.* $\beta_g \sim -(g - g^*)/2$ and $\beta_{M^2} \sim 0$. Thus, as $g \to g^*$ around a $c = 1$ fixed point, solving (6.15) for the ansatz $\Delta = \mathcal{F}_1\Delta(g)$ gives

$$\Delta(g) = \frac{K}{(g-g^*)^2} \Rightarrow \Delta = \frac{\mathcal{F}_1 K}{(g-g^*)^2}\,. \tag{6.16}$$

Here K is nothing but an undetermined integration constant. Choosing this constant to be $K = C(M^2 + \frac{n^2}{R^2})^2/\mathcal{F}_1$ makes the solution for Δ in (6.16) equivalent to that already given by (6.14). Thus the first of the two relations in (6.15) is a redundant constraint for Δ. Now let us solve the second relation in (6.15) by separation of variables, $a = \frac{1}{N}\,\tilde{a}(g, M, R)$, which (up to a constant factor) gives

$$a \sim \frac{1}{N}\,(g-g^*)^{\frac{Ng*^2}{Mg^2}\coth(\pi MR)\,\left(-\frac{1}{C^2}-\frac{e^{-C}}{C}-2e^{-C}\right)\,\tilde{l}^2}\; e^{\frac{N}{M}(\frac{1}{2}+\frac{g*}{g})\coth(\pi MR)\,\left(-\frac{1}{C^2}-\frac{e^{-C}}{C}-2e^{-C}\right)\,\tilde{l}^2}\,. \tag{6.17}$$

Here C is given by the spectrum (6.13). We see that (in accordance with [53]) the multiplicative prefactor a absorbs all the non-universal parameter dependence of the wave function and the universal part goes like $\Psi(l < \tilde{l}) \sim e^{-C\frac{l^2}{\tilde{l}^2}}$. Note that, the R dependence in the $\tilde{l}$ comes only through the Fourier modes of matter indicating *matter dependence*. Thus the full solution for the wave function at short distance (*i.e.* below Hubble scale) can be summarized to be

$$\Psi(l < \tilde{l}) = \frac{1}{N} \; (g - g^*)^{\frac{Ng*^2}{Mg^2} \coth(\pi MR) \; \left(\frac{1}{C^2} - \frac{e^{-C}}{C} - e^{-C}\right) \; \tilde{l}^2}$$
$$\times \, e^{\frac{N(g+g*)^2}{2Mg^2} \coth(\pi MR) \; \left(\frac{1}{C^2} - \frac{e^{-C}}{C} - e^{-C}\right) \; \tilde{l}^2} \; e^{-C \; l^2/\tilde{l}^2} \, , \tag{6.18}$$

where C is given by (6.13). Note that the nucleation probability given by this wave function is a mixture of the usual exponential suppression and the power law μ^δ (here $\mu \sim (g - g^*)$), exhibited by the inflating lower energy phase in the critical droplet fluctuation in $2D$ Liouville gravity, that tends to mimic the behavior of dynamical lattice Ising model [21].

The first few values of C computed in table above shows that for $n \geq 3$ the wave function indeed is a decaying wave function. In this regime, for odd n, $C \geq 0$ and the wave function is a purely decaying wave function. For even n, C is complex. Hence the wave function has a outgoing plane wave part along with a decaying part. Note that, only for $n = 1$, the wave function has a growing behavior (the usual Hartle-Hawking behavior) instead of a decaying one, along with an outgoing plane wave. However, n here are in one to one correspondence with the orders of the derivatives of l. For computations away from minisuperspace, indeed $n \geq 3$, and hence there are enough spectra giving a decaying solution. The growing solution can be suppressed by the choice of the boundary condition $\Psi(0) = \Psi_0 \neq 0$. Another point to be noted here is that, since for any value of n the wave function does not have any incoming plane wave part, one can say that the closed universes described by our wave function never re-collapses.

Such an exponentially decaying wave function can be interpreted to describe quantum creation of a closed inflationary universe filled with homogenous field $\hat{x}$ tunneling from a state of vanishing scale or *nothing*. To understand what kind of vacuum it tunnels to, let us consider the functional dependence of $\tilde{l}$ on $\hat{x}$ from (6.6). At the end of the tunneling, eventually at a scale $\tilde{l}(\hat{x})$, the universe in a state of large total energy (for example note that, in the standard chaotic inflation scenario, the initial energy density is of the order of M_p^4) $E \sim M$ (here $[M] = [l^2] = [\hat{x}^{-1}] = [E]$) will undergo a huge exponential expansion over a very short period of MQM time M^{-1} or due to a very small quantum fluctuation $\delta t \sim M^{-1}$. This is indeed happening for Dirichlet boundaries ($\delta\hat{x} \sim O(1/M) \to 0$). At the Hubble scale, to which the universe tunnels to, a very small quantum fluctuation sets an exponential growth in scale for a universe of huge energy. *Thus for $l < \tilde{l}$, (6.18) is the quantum wave function representing an one dimensional homogenous universe tunneling from 'nothing' to a state of chaotic inflation.*

Note that the Hartle-Hawking wave function computed away from the minisuperspace approximation, predicts 'tunneling to an inflationary vacuum from nothing', in a structurally similar way (apart from the fact that here $l = l(\phi(\tau))$ and not just $l = l_0(t)$) to the Linde-Vilenkin tunneling wave function in minisuperspace cosmological model

$$\Psi(l_0 < H^{-1}) \sim e^{-\frac{\pi}{2}H^2 l_0^2} . \tag{6.19}$$

This is unlike the usual growing behavior of the Hartle-Hawking wave function that makes predicting inflation to be impossible. One can argue that the boundary condition $\Psi(0) = \Psi_0 \neq 0$ chosen here may introduce arbitrariness in the initial condition of the universe. However, the Hartle-Hawking wave function $\Psi(l)$ is the wave function for the one point function with one puncture. The wave function for the $1D$ universes, in the true sense of the term (or according to traditional definition of the term), is the smooth one point function $\psi(l) = l\ \Psi(l)$ of the loop operator. This vanishes in the UV and hence has no arbitrariness in the initial condition. It has a growing and then rapidly decaying behavior capturing inflation.

Let us now argue for the uniqueness of the choice of our trial solution (6.9). As the WdW equation (5.26) is of infinite degree, in principle one needs an infinite dimensional initial condition to uniquely choose the solution. Here we argue that a gaussian profile of the initial ($l = 0$) scale fluctuation

$$\Psi(0 + \delta l) = \Psi_0 e^{-\Delta \delta l^2} , \tag{6.20}$$

fixes the initial values of all the derivatives

$$\partial_l^{2n+1}\Psi(0) = 0 , \quad \partial_l^{2n}\Psi(0) = \frac{(2n)!\Delta^n}{n!(-1)^n}\Psi_0 . \tag{6.21}$$

This provides the infinite dimensional initial condition to uniquely choose the wave function of the form $\Psi(l < \tilde{l}) \sim e^{-\Delta l^2}$ as the solution of (5.26).

Let us now explain the significance of the gaussian profile of the initial scale fluctuation. The density perturbation due to the scale fluctuation of the one dimensional universe we are considering, is given by

$$\frac{\delta\rho}{\rho} \sim \frac{\delta l}{l} = P_{l\to l+\delta l} = \frac{\Psi(l+\delta l)}{\Psi(l)} , \tag{6.22}$$

where we define $P_{l\to l+\delta l}$ to be the probability of finding an universe of scale l between l and $l+\delta l$. As we know, in an exponentially expanding universe, H^{-1} acts as a cut-off wavelength. The vacuum fluctuations with a wavelength greater than H^{-1} freezes as a classical field, producing density perturbations [19, 20]. Now using (6.18) in (6.22)

$$\frac{\delta l}{l} = e^{-C\frac{l^2}{\lambda^2}[2\frac{\delta l}{l}+(\frac{\delta l}{l})^2]} \simeq \left(1 - 2Cl^2/\lambda^2 \; \delta l/l + O(\delta l/l)^2\right), \tag{6.23}$$

where $\lambda = H^{-1}$. This gives

$$\frac{\delta l}{l} \sim \frac{\lambda^2}{\lambda^2 + 2Cl^2}. \tag{6.24}$$

Computing the quantity (6.22) for $l \gtrsim \lambda = H^{-1}$ we have

$$\frac{\delta \rho}{\rho} \sim \frac{\delta l}{l} \simeq \frac{\lambda^2}{2Cl^2} + O(\lambda^4/l^4). \tag{6.25}$$

This implies a *nearly nonflat* spectrum of density fluctuation at $l \gtrsim \lambda = H^{-1}$(*i.e.* at big bang). For example, for $\lambda \sim l^{1.01}$ (recall $l << 1$ at the initial moment), in order to achieve this bound on the wavelength,

$$\frac{\delta \rho}{\rho} \sim l^{0.02}. \tag{6.26}$$

This implies a scalar spectral index $n_s = 0.96$*.* Thus, presumably one could argue that the gaussian initial scale fluctuation profile that uniquely chooses the wave function is not an adhoc assumption, rather it physically arises from a special scale dependence of the density perturbations.

6.3 The large scale behavior: far future de Sitter

Let us now consider the large scale limit $l > \tilde{l}$. We would like to know whether the wave function in this region behaves like outgoing plane wave. This would be interesting cosmologically, as outgoing (incoming) plane waves in this region would represent expanding (contracting) global De-Sitter geometry in the far future [46]. However, universe at large scale is a classical object. So the semiclassical wave function extrapolated to far future is able to capture such expanding geometries. The inability of the semiclassical wave function is in capturing the inhomogeneities that sets in due to quantum fluctuation in the early universe.

In the large scale limit $l > \tilde{l}$, the WdW equation (5.26) can be written as

$$\Big[A\cosh\big(\tilde{l}\frac{d}{dl}\big) + B\lambda_n \sinh\big(\tilde{l}\frac{d}{dl}\big) - \frac{\mathcal{F}}{2}\frac{d}{dl} - A\Big]\Psi(l) = 0. \tag{6.27}$$

Here we have rewritten (5.20) appropriately for the wave function proportional to a plane wave in l space and kept only the contribution from the anomalous dimension that is significant in the large scale limit. Clearly the wave function at large scale can assume (outgoing and incoming) plane wave solutions of the form $\Psi(l) = e^{\pm i\Delta l}$ with

$$\mathcal{F} = N\frac{\partial \Delta}{\partial N} + \frac{g}{2}\frac{\partial \Delta}{\partial g} = 0 \tag{6.28}$$

Here we have used the fact that around $c = 1$ fixed point, the betafunctions behaves as $\beta_g \sim -g/2$, $\beta_{M^2} \sim 0$, with $Ng^2 = 1/g_{st} = const$. This forces $\mathcal{F}$ to vanish automatically. The corresponding spectrum of Dirichlet boundaries at large scale (the one dimensional universes with homogenous field) is then given by

$$A\cosh(i\tilde{l}\ \Delta) \pm B\lambda_n \sinh(i\tilde{l}\ \Delta) - A = 0\,. \tag{6.29}$$

This implies the wave function at large scale is an outgoing plane wave (or a far future expanding De-Sitter geometry) with the spectrum

$$\Delta_{m,n} = \frac{2m\pi}{\tilde{l}} = \frac{2m\pi}{g}(M^2 + n^2/R^2)\,, \quad m, n = 0, \pm 1, \pm 2, \ldots \tag{6.30}$$

Again, the non-universal part of the spectrum is only due to Fourier modes of matter and factorizes in the wave function. The universal part of the spectrum goes like $\Delta_m = \frac{2m\pi M^2}{g}$.

6.4 The de Sitter minisuperspace?

Here we will show that suppressing the effect of higher order derivatives with respect to the scale in (5.26) leads to a de Sitter minisuperspace type of WdW cosmology, namely, a global de Sitter in the far past instead of a tunneling wave function for an inflationary universe. The far future dynamics of global de Sitter is unchanged, as would be expected.

For this purpose, let us consider the expansion of the hyperbolic differential operators in (5.26) using $\tilde{l}$ as the expansion parameter. This is consistent as $\tilde{l} << 1$ around a $c = 1$ fixed point. We also consider $l \lesssim \tilde{l}$ and $B/A \sim O(1/M) \to 0$. The denominator of the fractional polynomial in l in the equation (5.26) can be expanded in powers of $l/\tilde{l}$. Thus keeping up to $O(l/\tilde{l})^2$ and second order derivatives in l, (5.26) can be rearranged as

$$\Big[- l^2\frac{d^2}{dl^2} + \frac{\mathcal{F}_2}{A\tilde{l}^2}\, l\frac{d}{dl} - \frac{2}{\tilde{l}^2}\Big(1 - \frac{\mathcal{F}_1}{A}\Big)l^2\Big]\Psi(l) = 0\,. \tag{6.31}$$

This is structurally very similar to a the minisuperspace WdW equation. To make contact with all genus minisuperspace WdW equation, one can in principle allow more terms in the potential, say l^4 term, by truncating the fractional polynomial potential in (5.26) at orders higher than $O(l/\tilde{l})^2$. We identify the factor ordering ambiguity p and the world sheet cosmological constant μ to be

$$
\begin{aligned}
p &= -\frac{\mathcal{F}_2}{A\tilde{l}^2} = \Big(2e^{-C} + \frac{e^{-C}}{C} + \frac{1}{C^2}\Big)\,, \\
4\mu &= -\frac{2}{\tilde{l}^2}\Big(1 - \frac{\mathcal{F}_1}{A}\Big) = -\frac{2}{\tilde{l}^2}\Big(1 + 2Ce^{-C} + \frac{1}{C}\Big) < 0\,,
\end{aligned}
\tag{6.32}
$$

with C given by (6.13). However, the De-Sitter sign of the cosmological constant $4\mu < 0$ clearly shows that the WdW equation derived with all modes of φ turned on (*i.e.* derived with N^2 quantum mechanical degrees of freedom) but truncated to space with quadratic scale derivatives is a De-Sitter minisuperspace WdW equation like that of a supercritical theory. This is different from the usual AdS minisuperspace WdW equation (2.4) or (4.7) of the (non)critical theory derived with zero mode truncation of dilaton φ_0 (*i.e.* with N quantum mechanical degrees of freedom) in a quadratic space of scale derivatives. This indirectly indicates a growth in space-time directions. The wave function given by (6.31) behaves as the modified Bessel function

$$
\Psi(l) \sim l^{\frac{1}{2}(p+1)} J_{-\frac{1}{2}(p+1)}(2\sqrt{-\mu}l)\,,
\tag{6.33}
$$

and thus would go like the De-Sitter minisuperspace wave function $J_0(\sqrt{-\mu}l)$ at $p = -1$. This represents a global de Sitter in the far past [46].

Recall that the factor ordering ambiguity in the minisuperspace WdW equation (2.4) in Liouville background is $p = -1$. This makes the wave function purely Bessel function and behaves as plane wave at $l \to 0$ rather than a tunneling wave function. Actually the solutions to (2.4) are different Bessel or Hankel functions depending on appropriate boundary conditions and the sign of μ. They give rise to different classical geometries corresponding to homogenous FRW universes. For example, Bessel $K_0(2\sqrt{-\mu}l)$ gives rise to AdS geometry ($\mu > 0$) corresponding to big bang and big crunch while Bessel $J_0(2\sqrt{-\mu}l)$ and Hankel $H_0(2\sqrt{-\mu}l)$ can be shown to give global de Sitter geometry ($\mu < 0$) in the far past and far future respectively [46]. These solutions are unable to capture any kind of inflationary scenario that introduces large inhomogeneities with evolution of time. In our large N RG, we see that we have a more general factor ordering ambiguity that can admit other values too. For $p = -1$ we recover the structure of the usual minisuperspace WdW in MQM.

However, the quantum wave function in the $l > \tilde{l}$ regime for Dirichlet boundaries has similar behavior as the minisuperspace wave function extrapolated to the large scale. For example the $l > \tilde{l}$ solution of (6.27), outgoing plane waves characterizing a far future de Sitter, can be obtained from the large scale extrapolation of the minisuperspace wave function, a Bessel function. Thus the far future solutions for universe with homogenous field can be qualitatively read from the minisuperspace wave function. presumably this is because the inhomogeneities at the large scale are captured more successfully by the dynamics of the nonzero modes of matter that is not dealt by the Dirichlet boundaries. Perhaps one needs to learn this dynamics from Neumann boundaries (D-1 branes) in $2D$ string theory.

Let us make a brief comment here regarding the magnitude of the de-Sitter cosmological constant as captured by large N RG. From (6.32) we observe

$$\Lambda_{dS} = 2H^2\Big(1 + 2Ce^{-C} + \frac{1}{C}\Big) > 0\,. \tag{6.34}$$

However, apparently the magnitude is not small due a large H (small g^*) around a $c = 1$ fixed point, though along the world sheet RG the system flows away from such a fixed point to a supercritical regime and we do not know eventually to what value of H (or g) it settles to. Nevertheless, as we will see in the last section, nonlocal processes like emission of baby universes self-tunes Λ_{dS} to a small positive value.

7 The baby universes

Let us now discuss the nonlinear part of the WdW constraint that connects the different topological sectors in the superspace. The cosmological processes of topology change, such as creation of a universe from nothing, or more generally loop splitting due to reconnection of intersecting strings giving birth to baby universe [34] or evolution of cosmic strings [71], connect different topological sectors of superspace (each such superspace sector being characterized by all metrics having the same topology). This suggests that the Wheeler-de Witt operator could be modified by adding a local operator that has matrix elements between different superspace sectors. As suggested by Vilenkin [4], this can be schematically written as

$$\mathcal{H}\psi_n(l) + \sum_{n'\neq n} \int [dl']\Delta_{nn'}(l,l')\psi_{n'}(l) = 0\,, \tag{7.1}$$

where l is the superspace variable (here the length of the loop) and n labels the topological sector (here the number of disconnected loops or the occupation number of the closed strings). Considering topology change to be a local process, it can be assumed that $\Delta_{nn'}(l,l')$ vanish unless l and l' are obtained from each other by changing topological relations at a single point, *i.e.* $\Delta_{nn'}(l,l') \neq 0$ for $n' = n \pm 1$. Knowing the functional form of $\Delta_{nn'}(l,l')$ needs to deal with string interaction vertices at the full quantum level. Even a qualitative knowledge of its functional form in the $2D$ case will be immensely helpful in understanding the topology changing amplitudes in the higher dimensions.

7.1 Nonlinear part of modified WdW equation

Here we will derive the emergence of the topology changing amplitudes from the nonlinearity in the WdW equation captured by large N RG . It is argued in [36] that such a nonlinearity in the WdW equation is essential in the matrix model description of $2D$ gravity. However, there seems to be no control over the numbers and sizes of the baby universes emitted or absorbed. Here we see that the world sheet RG is very special in this respect that it gives rise to baby universes of vanishing size with amplitudes suppressed as the higher order corrections in $O(1/N)$ to WdW equation. This implies that such processes are suppressed at large scale and at weak string coupling. The suppression of the amplitudes comes as a virtue of the vanishing size of the baby universes. It will be extremely interesting to understand the underlying phenomena that enables the world sheet RG in the large N language to control the size of the baby universes.

Let us recall (5.18) and consider the multi-trace or wormhole terms represented by the ellipses in the equation. In matrix quantum mechanics they are redundant operators. Their effects are lower by relative orders of $O(1/N)$ compared to the leading contribution. To capture their corrections to the single topology part of the WdW equation, let us rewrite the expansion $\tilde{\Sigma}^{-1}$ from (3.16) as

$$
\begin{aligned}
&\Big\langle \tilde{\Sigma}^{-1}\mathrm{Tr}\frac{e^{l\phi(\hat{x})}}{N}\Big\rangle_{\eta_i} \sim \Big\langle \mathrm{Tr}\frac{e^{l\phi(\hat{x})}}{N}\Big\rangle_{\eta_i} \\
&+\Big\langle\Big[\int dt\int dl'\delta(l')\Big(g\ F_{g1}\frac{\partial}{\partial l'}+g^2\ F_{g2}\frac{\partial^2}{\partial l'^2}+g^3\ F_{g3}\frac{\partial^3}{\partial l'^3}+...\Big)\mathrm{Tr}e^{l'\phi(t)} \\
&+\int dl'dl''\ \delta(l')\delta(l'')\Big\{\int dt'dt''\Big(g^2\ F_{g1}^2\frac{\partial}{\partial l'}\frac{\partial}{\partial l''}+2g^3F_{g1}F_{g2}\frac{\partial}{\partial l'}\frac{\partial^2}{\partial l''^2}+...\Big) \\
&-\int dt' g^2\ \hat{F}_{g2}\frac{1}{l'l''}\frac{\partial^2}{\partial t'^2}\Big\}\mathrm{Tr}e^{l'\phi(t')}\mathrm{Tr}e^{l''\phi(t'')} \\
&+...\Big]\mathrm{Tr}\frac{e^{l\phi(\hat{x})}}{N}\Big\rangle_{\eta_i}\,.
\end{aligned}
\tag{7.2}
$$

Here F_{gi}-s are hyperbolic functions given by (3.17). Using the expansion in (7.2) into (5.15) and recalling (5.8), the inclusion of the topology changing amplitudes in the beyond minisuperspace WdW can be arranged as

$$
\mathcal{H}\psi_1(l)+\sum_{n\neq 0,-1}\prod_{i,j=1}^{n}\int dl'_j\ \int dt_i\Big[\Delta_{1,n+1}(l;l'_j)\psi_{n+1}(l;l'_j,t_i)+\tilde{\Delta}_{1,n+2}(l;l'_j,t_i)\psi_{n+2}(l;l'_j,t_i)\Big]=0\,,
\tag{7.3}
$$

where $\mathcal{H}$ represents the WdW operator acting on a given topological sector. Let us call this part single topology WdW operator. Here $\mathcal{H}$ acts on wave function $\psi_1(l)$ corresponding to a single loop and is given by (5.19)

$$\mathcal{H} = \left(\frac{1}{l^2} - \frac{1}{l}\frac{\partial}{\partial l}\right)\left(N\frac{\partial}{\partial N} - \beta_{\eta_i}\frac{\partial}{\partial \eta_i} + 1\right) + A\cosh\left(\tilde{l}\frac{\partial}{\partial l}\right) + B\lambda_n \sinh\left(\tilde{l}\frac{\partial}{\partial l}\right). \tag{7.4}$$

ψ_n-s represent n-point functions or the n-loop amplitudes

$$\psi_n(l_1, l_2, ..l_n; t_1, t_2, ..t_n) = \left\langle \mathrm{Tr}\frac{e^{l_1\phi(t_1)}}{N}\mathrm{Tr}\frac{e^{l_2\phi(t_2)}}{N}...\mathrm{Tr}\frac{e^{l_n\phi(t_n)}}{N}\right\rangle. \tag{7.5}$$

To have a flavor about the nature of the matrix elements between the different topological sectors, let us write down first few matrix elements as differential operators

$$\begin{aligned}
&\Delta_{1,2}(l;l') = N\delta(l')\sum_{m=1}\left(g^m\ F_{gm}\frac{\partial^m}{\partial l'^m}\right)\left(A\cosh\left(\tilde{l}\frac{\partial}{\partial l}\right) + B\lambda_n\sinh\left(\tilde{l}\frac{\partial}{\partial l}\right)\right),\\
&\Delta_{1,3}(l;l',l'') = N^2\delta(l')\delta(l'')\sum_{m=1}\left(g^m\ F_{gm}\frac{\partial^m}{\partial l'^m}\right)\sum_{k=1}\left(g^k\ F_{gk}\frac{\partial^k}{\partial l''^k}\right)\\
&\times\left(A\cosh\left(\tilde{l}\frac{\partial}{\partial l}\right) + B\lambda_n\sinh\left(\tilde{l}\frac{\partial}{\partial l}\right)\right),\\
&\tilde{\Delta}_{1,3}(l;l',l'',t') = -N^2\delta(l')\delta(l'')g^2\ \hat{F}_{g2}\frac{1}{l'l''}\frac{\partial^2}{\partial t'^2}\left(A\cosh\left(\tilde{l}\frac{\partial}{\partial l}\right) + B\lambda_n\sinh\left(\tilde{l}\frac{\partial}{\partial l}\right)\right).
\end{aligned} \tag{7.6}$$

They represent branching of a loop into n loops out of which $(n-1)$ have vanishing radius. Thus the $(n-1)$ loops are emitted (absorbed) as 'baby universes' from (by) the original one. As differential operators the dependence on the length of the mother loop and the daughter loops factorizes. However, their interacting nature is unveiled by plugging in the scale dependence of the n-loop wave function. It will be certainly interesting to solve (7.3) by some numeric ansatz for the scale dependence of ψ_n by and get the scale dependence of the topology changing amplitudes.

To get a clear idea about the relative order of these multi-trace (wormhole) contributions in $O(1/N)$, one can absorb the extra N dependence in each amplitude by rescaling the length of the vanishing loops as $l' \to \tilde{l}' = Nl' \to 0$. This arranges the multi-trace or wormhole terms in relative strength of $O(1/N)$ compared to the single trace terms. For example, The correction due to $\Delta_{1,2}$ begins from a relative $O(1/N)$ compared to the single topology part of the WdW equation. Similarly, the correction due to $\tilde{\Delta}_{1,3}$ is of relative $O(1/N^2)$ and the correction due to $\Delta_{1,3}$ begins from a relative $O(1/N^2)$ compared to the single topology WdW equation. Note that, at a given order the contribution from kinetic term of the matrix quantum mechanics has much larger contribution than that coming from the quadratic term or the cubic interaction vertices.

It will be extremely interesting to understand the physics in the world sheet RG that controls the baby universes to have vanishing size and hence being suppressed by $O(1/N)$. In the large

N RG, from the point of view of the matrix quantum mechanics, this is very closely attached to the matrix nature of $\phi(t)$ and hence to the involvement of all the N^2 quantum mechanical degrees of freedom in the dynamics. This indicates that perhaps a proper understanding of the physics of these N^2 quantum mechanical degrees of freedom in the strong coupling and at small scale is a key step towards real cosmology.

7.2 Self-tuning universe

Here we will show how the sub-leading contribution to the linear part of the modified WdW equation, namely the contribution coming from the formation of one baby universe, self-tunes the cosmological constant to a small positive value. As pointed out in [64], the mechanism of self tuning of the cosmological constant by uncontrollable emission of baby universes [34] has been studied extensively in the literature. However, the main problem is to understand the issues like the locality of the coupling of the baby universes to the parent universe. As we have discussed above, in large N world sheet RG a controlled emission of baby universes of vanishing size automatically emerges from the nonlinear part of the modified WdW equation (7.3). These events have local coupling with the parent universe by the virtue of the vanishing size of the baby universes. Below we will derive the leading contribution of such event from the nonlinear part of the modified WdW equation (7.3) and its role in self tuning the de Sitter cosmological constant (6.34).

Considering the leading contribution from the emission of one baby universe, the nonlinear modified WdW equation takes the form

$$\Big(\frac{1}{l^2} - \frac{1}{l}\frac{\partial}{\partial l}\Big)\Big(N\frac{\partial}{\partial N} - \beta_{\eta_i}\frac{\partial}{\partial \eta_i} + 1\Big)\psi_1(l,\hat{x}) + \Big(A\cosh(\tilde{l}\frac{\partial}{\partial l}) + B\lambda_n \sinh(\tilde{l}\frac{\partial}{\partial l})\Big)$$
$$\Big(\psi_1(l,\hat{x}) + \frac{1}{N}\,\frac{g\coth(\pi MR)}{2M}\int dl'\delta(l')\frac{\partial}{\partial l'}\int dt'\psi_2(l,l';\hat{x},t')\Big) = 0\,. \tag{7.7}$$

However, integrating by parts the $O(1/N)$ term can be simplified as

$$\int dl'\delta(l')\frac{\partial}{\partial l'}\int dt'\psi_2(l,l';\hat{x},t') = -\int dl'\Theta(l')\int dt'\psi_2\,, \tag{7.8}$$

where the boundary term $\delta(l')\int dt'\psi_2(l,l';\hat{x},t')$ vanishes ψ_2 being a continuous function of l' and $\Theta(l')$ represents the Heaviside step function. Performing the l' integration for $l' > 0$ (7.8) becomes proportional to

$$\int dl'\delta(l')\frac{\partial}{\partial l'}\int dt'\psi_2(l,l';\hat{x},t') \sim -e^{\infty}\int^t \frac{dt'}{\lambda(t')}\psi_1\,, \tag{7.9}$$

where we have assumed all the N eigenvalues of $\phi(t)$ are same, given by $\lambda(t)$, and the factor of order e^{∞} comes from the $l' \to \infty$ end of the definite integral. Considering the periodic eigenvalue to be $cos(\frac{n}{R}t)$, such a contribution will modify the de Sitter cosmological constant (6.34) by modifying the constant $A \to A(1-\sigma)$ to be

$$\Lambda'_{dS} = 2H^2\Big(1 - \frac{\mathcal{F}_1}{A(1-\sigma)}\Big) = 2H^2\Big(1 + \frac{2Ce^{-C} + \frac{1}{C}}{1-\sigma}\Big)\,, \tag{7.10}$$

where σ is given by

$$\begin{aligned} \sigma &= \frac{1}{N}\frac{gR}{2nM}\coth(\pi MR)\int^t \frac{dt'}{\lambda(t')}\, e^{\infty} \\ &= \frac{1}{N}\frac{gR}{2nM}\coth(\pi MR)\sec(\frac{n}{R}t)\tan(\frac{n}{R}t)\, e^{\infty} \gg 1\,. \end{aligned} \tag{7.11}$$

Thus the de Sitter cosmological constant becomes

$$\Lambda_{dS} \to \Lambda'_{dS} \sim 2H^2\Big(1 - \frac{2Ce^{-C} + \frac{1}{C}}{\sigma}\Big)\,, \tag{7.12}$$

where it is possible to get $\frac{2Ce^{-C}+\frac{1}{C}}{\sigma} \lesssim 1$ for small C as the numerator diverges in inverse power in C whereas the denominator diverges exponentially. Thus irrespective of the magnitude of H, for small C the sub-leading effect of the emission of a single baby universe can achieve the desired self tuning $\Lambda_{dS} \to \Lambda'_{dS} \gtrsim 0$. It will be extremely interesting to capture the phenomena from some kind of nonlocal world sheet action (see the world sheet methods in [25, 26]).

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